Analysis Limits Test 2013-2014 $\frac{90}{100}$ → 92 95 Jason Huang is going to dominate (squeeze?) this test

Deggeller/Hahn **Date** 4/17/14 **Period** F

NO CALCULATOR

Some potentially helpful formulas:

$$\sum_{a=1}^{n} a^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{a=1}^{n} a^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Multiple Choice. [4 pts each]

1. The first four terms of the sequence $\left\{\left(1-\frac{1}{2^n}\right)^2\right\}$ (starting with n=1) are:

A) $\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16}$ B) $\frac{1}{4},\frac{9}{16},\frac{49}{64},\frac{225}{256}$ C) $\frac{1}{2},\frac{3}{4},\frac{5}{6},\frac{7}{8}$ D) $\frac{1}{4},\frac{9}{16},\frac{25}{36},\frac{49}{64}$ E) none of these

2. The infinite power series $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$ is equivalent to

A) ln x B) e^x C) sin x D) 0 E) the series diverges

3. Which of the following is a subsequence of $\{n^2+(-1)^n\}$? 0, 5, 8, 17, 24

A) $\{n^2\}$ B) $\{-n^2\}$ C) $\{n(n+1)\}$ D) $\{(2n+1)^2-1\}$ E) None of These

4. The **sequence** $\left\{\sqrt{\frac{16n^5-9n^3+2n}{2n^5+7n^4-3n^2}}\right\}$ converges to a value of _____.

A) $\sqrt{\frac{3}{2}}$ B) 0 C) 4 D) 8 E) $2\sqrt{2}$

5. Which of the following **sequences** converge?

I. $\left\{\frac{1}{n}\right\}$ II. $\left\{(2)^{-n}\right\}$ III. $\{n\}$ IV. $\left\{\frac{1}{n^2}\right\}$ V. $\left\{\left(\frac{4}{3}\right)^n\right\}$

A) I, II, and IV B) I, II, III, and V C) II and IV D) II only

/p

6. For each **series**, write "C" if it converges, and "D" if it diverges. **[3 pts each]**

a) $\sum_{n=1}^{\infty} .95^n$ C

b) $\sum_{n=1}^{\infty} 1.05^n$ D

c) $\sum_{n=1}^{\infty} \frac{1}{1.05^n}$ C

d) $\sum_{n=1}^{\infty} \frac{1}{n^{1.05}}$ C

this is correct

e) $\sum_{n=1}^{\infty} \frac{1}{n^{-1.05}}$ D

f) $\sum_{n=1}^{\infty} n^{.95}$ D

g) $\sum_{n=1}^{\infty} \frac{1}{n^{.95}}$ D

h) $\sum_{n=1}^{\infty} \sin(n)$ D

Free Response

7. For the function $f(x) = 4x^2 + 3x$, use the difference quotient (no calculus allowed) to find an expression for the slope of the tangent line at **any** value of x. **[6 pts]**

$$\frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} = \left(4\left(x^2 + \frac{2x}{n} + \frac{1}{n^2}\right) + 3x + \frac{3}{n} - 4x^2 - 3x\right)n$$

$$= \left(\cancel{4x^2} + \frac{8x}{n} + \frac{4}{n^2} + \cancel{3x} + \frac{3}{n} - \cancel{4x^2} - \cancel{3x}\right)n$$

$$= 8x + 3 + \cancel{\frac{4}{n}}$$

$$= \boxed{8x + 3}$$

/3

8. Does each series converge or diverge? Justify your answer with a proof. Name the method you use in your work. **[6 pts each]**

a) $\sum_{n=0}^{\infty}\frac{1}{4+5^n} < \sum_{n=0} \frac{1}{5^n}$

$\frac{1}{5}+\frac{1}{9}+\frac{1}{29}+\frac{1}{129}+\cdots$ is less than

$\frac{1}{1}+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots \rightarrow \frac{1}{5}<1, \frac{1}{9}<\frac{1}{5}, \frac{1}{29}<\frac{1}{25}$, so on..

$\frac{1}{5^n}$ is convergent, so this series is convergent.

Comparison test

b) $\sum_{n=0}^{\infty}\frac{3n!(-1)^n}{4n!-2}$ Divergent

① Alternating (strictly)

②

-3 test ~~ndre~~ used?

③ $\lim \rightarrow \infty = \frac{3}{4}$ ×

c) $\sum_{n=1}^{\infty}\frac{2^n(n+1)}{5^n} < \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$

$\frac{4}{5}+\frac{12}{25}+\frac{32}{125}+\frac{80}{625}+\cdots$ is less than

$\frac{4}{5}+\frac{16}{25}+\frac{64}{125}+\frac{256}{625}+\cdots \rightarrow \frac{4}{5}=\frac{4}{5}, \frac{12}{25}<\frac{16}{25}, \frac{32}{125}<\frac{64}{125}$, so on...

$\left(\frac{4}{5}\right)^n$ is convergent, so this series is convergent.

Comparison test

d) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{3n^2+5}$ Convergent

① Strictly alternating

② $\cancel{\frac{(-1)^{n+1}(n+1)}{3(n+1)^2+5}}$ $\quad (-1)(n+1)(3n^2+$

$\cancel{\frac{(-1)^n n}{3n^2+5}}$ $\quad = (3(n^2+2n+1)+5)$

$\frac{(-1)^{n+1}(n+1)}{3(n^2+2n+1)+5} < \frac{(n)}{3n^2+5}$

$-(3n^3+3n^2+5n+5) < 3n^3+6n^2+3n+5$

$6n^3+9n^2+8n+10>0$ ✓

③ $\lim \rightarrow \infty = 0$

Alternating series test

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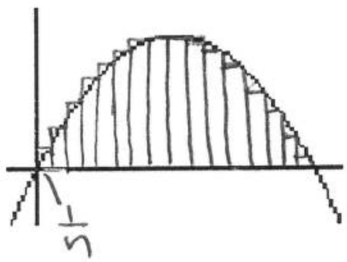
9. Show your use of sequences and/or series to find the area between the graph of $y = x - x^2$ and the x-axis. **[7 pts]**



$x - x^2 = 0 \quad x(1-x) = 0 \quad x = 0, 1$

$$\frac{1}{n}\left(\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\cdots+\frac{n}{n}\right)$$

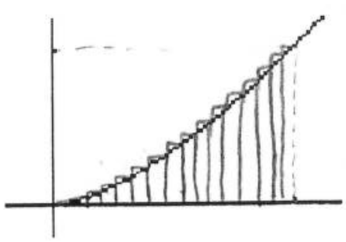
$$-\frac{1}{n}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right)$$

$$=\frac{1}{n}\left(\frac{n(n+1)}{2n}\right)-\frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6n^2}\right)$$

$$=\frac{n^2}{2n^2}+\cancel{\frac{n}{2n^2}}-\frac{2n^3}{6n^3}-\cancel{\frac{3n^2}{6n^3}}-\cancel{\frac{n}{6n^3}}$$

$$=\frac{1}{2}-\frac{1}{3}=\boxed{\frac{1}{6}}$$

10. Show your use of sequences and/or series to find the volume created when the graph of $y = x^{\frac{3}{2}}$ bounded by the x-axis and the line $x = 1$ is rotated around the. x-axis. **[7 pts]**



$$\frac{\pi}{n}\left(\left(\left(\frac{1}{n}\right)^{\frac{3}{2}}\right)^2+\left(\left(\frac{2}{n}\right)^{\frac{3}{2}}\right)^2+\left(\left(\frac{3}{n}\right)^{\frac{3}{2}}\right)^2+\left(\left(\frac{4}{n}\right)^{\frac{3}{2}}\right)^2+\cdots+\left(\left(\frac{n}{n}\right)^{\frac{3}{2}}\right)^2\right)$$

$$=\frac{\pi}{n}\left(\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\left(\frac{3}{n}\right)^3+\cdots+\left(\frac{n}{n}\right)^3\right)$$

$$=\frac{\pi}{n}\left(\frac{(n(n+1))^2}{4n^3}\right)=\boxed{\frac{\pi}{4}}$$

11. The function, $f(x)=\tan^{-1}(x)$, over the interval x: [-1, 1] can be modeled by the power series

$$P(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+......$$

a) Use sigma notation to express this infinite series **[4 pts]**

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)\cancel{!}}(-1)^n$$

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b) Write a sigma notation expression that represents the sum of the first 8 terms of P(1). **[3 pts]**

$$\sum_{n=0}^{7}\frac{1}{(2n+1)!}(-1)^n$$

c) Evaluate the infinite sum $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.....$ exactly. **[2 pts]**

$$\tan^{-1}(1)=\boxed{\frac{\pi}{4}}$$

d) Sadly the power series P(x) above does not accurately model $\tan^{-1}(x)$ for **all** values of x. Why would it not work for x = 5? **[3 pts]**

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The function can only accurately model $\tan^{-1}(x)$ for values between -1 and 1.

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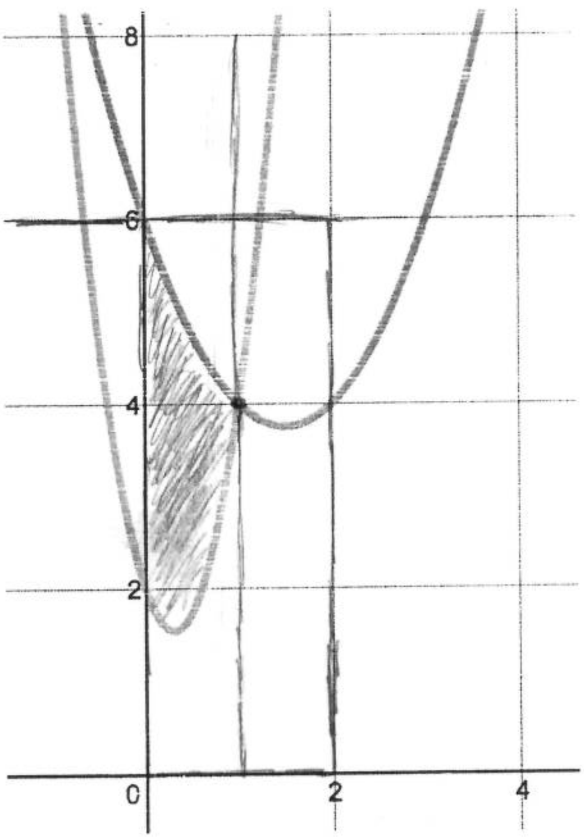
Extra Credit: (this question is worth only 1 points. Do not attempt this problem if you're not finished with the rest of the test first).

Use summations of infinite sequences to find the area of the region bounded between the curves $f(x)=x^2-3x+6$ and $g(x)=5x^2-3x+2$, for values of $0 \le x \le 1$. The graph of the functions is provided below.



Area under f(x):

$$\left|\frac{1}{n}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots\left(\frac{n}{n}\right)^2-3\left(\left(\frac{1}{n}\right)+\left(\frac{2}{n}\right)+\left(\frac{3}{n}\right)+\cdots+\left(\frac{n}{n}\right)\right)+6n\right)\right|$$

$$=\left|\frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6n^2}-\frac{3n(n+1)}{2n}+6n\right)\right|=\left|\frac{1}{3}-\frac{3}{2}+6\right|=\frac{29}{6}$$

Area under g(x):

$$\left|\frac{1}{n}\left(5\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right)-3\left(\left(\frac{1}{n}\right)+\left(\frac{2}{n}\right)+\left(\frac{3}{n}\right)+\cdots+\left(\frac{n}{n}\right)\right)+2n\right)\right|$$

$$\left|\frac{1}{n}\left(\frac{5n(n+1)(2n+1)}{6n^2}-\frac{3n(n+1)}{2n}+2n\right)\right|=\left|\frac{5}{3}-\frac{3}{2}+2\right|=\frac{13}{6}$$

$$\frac{29}{6}-\frac{13}{6}=\boxed{\frac{8}{3}}$$