Math Analysis Honors – Deggeller / Hahn
Midterm 5: GAtM 2016-17
No Calculators!

Did you shear the good news?! Hannah Kim
Period: 6

82/86

Multiple Choice:

1. The number 0.123456 corresponds to the coordinate point (0.135, 0.246), and a similar correspondence can be drawn for any number between 0 and 1. This helps to prove that which two following infinite sets are of equal size?

a) {rational numbers} and {irrational numbers}

b) # of points on a line segment and # of points on a unit square

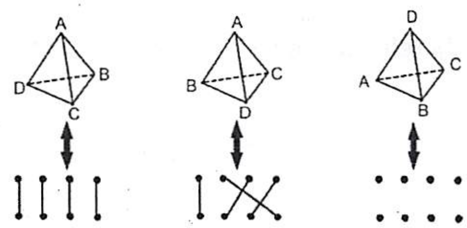
c) # of points on a line segment and # of points on a line

d) {positive rational numbers} and {natural numbers}

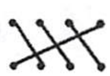
e) {numbers between 0 and 1} and {numbers between 0 and 4}

2. Which of the following is NOT a characteristic of groups?

a) Invertibility b) Identity c) Commutativity d) Associativity e) Closure



3. The 4-post snap group is isomorphic to the reflection group of a tetrahedron, and a one-to-one correspondence can be drawn between elements of the two groups. In the figure above, a correspondence is shown for some of the elements. Which answer choice shows the missing snap element?

a)  b)  c)  d)  e) 

4. On the complex number plane, if z is in quadrant 3, then -iz will be in quadrant

a) 1 only b) 2 only c) 3 only d) 4 only e) it depends on the exact value of z

$90^\circ \cdot 3$
270

5. The group generated by the multiplication of complex numbers is isomorphic to the group generated by

a) rotations and shears b) stretches and shears c) stretches and rotations

d) rotations and reflections e) dilations and rotations

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

-3

For Questions 6-8, refer to the following matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

6. What would be a possible eigenvector (with associated eigenvalue) of matrix A?

a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (with eigenvalue of 1) b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (with eigenvalue of 3) c) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (with eigenvalue of 1)

d) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (with eigenvalue of 3) e) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (with eigenvalue of 3)

$\begin{bmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix}$ $(1-\lambda)^2 = 0$ $1^2 + \lambda^2 - 2\lambda = (\lambda-1)^2$ $\lambda = 1$ $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \times 0$

7. What is the order of the group generated by Matrix C?

a) 2 b) 4 c) 6 d) 12 e) 24

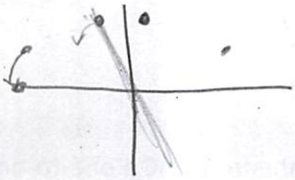
$cis\,\frac{\pi}{6}$

8. Matric CB is a

a) rotation by 60 degrees b) rotation by 120 degrees

c) reflection over the line y = xtan(210) d) reflection over the line y = xtan(120)

e) reflection over the line y = xtan(105)



$80 \to 100 \to 130$

9. If $z = 3\,cis\frac{\pi}{6}$, then

a) $Re(z^{25}) < 0$ b) $Im(z^{25}) = 0$ c) $Im(z^{25}) > 0$ d) $Arg(z^{25}) = 0$ e) $\bar{z} = z$

$3^{25}\,cis\,\frac{25\pi}{6}$

$3^{25}\,cis\,\frac{\pi}{6}$

10. Matrix $\begin{bmatrix} 3 & 2 \\ 7 & x \end{bmatrix}$ is used to transform a unit square into a parallelogram with an area of 35 square units. Which is a possible value of x?

a) 35 b) 21/2 c) 7 d) 49/3 e) -6

$3x - 14 = 35$

$3x = 49$

$x = \frac{49}{3} =$

Free Response:

11. Name 3 different groups that each have 40 members, but where groups 1 and 2 are isomorphic to each other, and group 3 is not isomorphic to the first two. [2 pts each]

Group 1: rotation group of 40-sided polygon

Group 2 (Isomorphic to Group 1): group generated by matrix $\begin{bmatrix} \cos\frac{\pi}{20} & -\sin\frac{\pi}{20} \\ \sin\frac{\pi}{20} & \cos\frac{\pi}{20} \end{bmatrix}$

Group 3 (not isomorphic to G1 or G2): reflection group of 20-sided polygon

12. We want to use matrices to transform the point (-2, 5) to (4, 5).

a) What matrix could be used to <u>**shear**</u> the point (-2, 5) into (4,5) ? [3]

$$\begin{bmatrix} 1 & \frac{6}{5} \\ 0 & 1 \end{bmatrix}$$

b) Name a matrix that could map (-2,5) to (4,5) by first doing a horizontal translation (glide) and then a reflection over the y-axis. Your answer should be a single matrix. [3]

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

c) Explain why it is impossible to do a rotation around the origin to map (-2,5) into (4,5). [2]

a rotation around the origin would accomplish nothing unless distance from the origin was also changed, as (-2, 5) is $\sqrt{29}$ units away from the origin, while (4,5) is $\sqrt{41}$ units away from the origin.

13. Consider the infinite set of numbers N defined as "The integer powers of 10" (e.g. $10^2, 10^{-3}$.....)

a) The set of numbers is not a group under addition. Which of the four group requirements does it fail to meet (name all that it fails). [4]

- invertibility (no negative numbers, can't go back to 0)
- identity (0 is not an integer power of 10; no identity)
- closure ($10^2 + 10^1 = 110$, which isn't w/in the group)

b) Is the set of numbers a group under multiplication? yes [2]

$10^2 \cdot 10^{-2} = 10^0$ $\quad 10^3 \cdot 10^{40} = 10^{43}$

$10^0 = I$ closed ✓
inv: 1, -1
ass ✓

c) Is group N the same size as the set of rational numbers? Briefly explain why. [3]

yes

integers moving across

rational numbers

→	1	2	3	4	5	6
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$

← each of these can be assigned a corresponding integer, in a diagonal fashion $\frac{1}{2}/\frac{3}{5}/\frac{4}{6}/\frac{7}{8}/\frac{9}{10}$ (I numbered some and was too lazy to do the others), but due to repeats, rational < integers. Integers, however in this direction → show that integers < rational. Integers are the value here $10^{\square}$. int < rational, rational < int, therefore rat = int, therefore N = rat.

← this is just a cross section
negative numbers continue to the left side.

14. The "Herreshoff Recipe" we learned for decomposing a matrix fails for Matrix M below because the upper left hand corner is 0. One way around this is to begin by reflecting the matrix over the line y=x to begin with. Let's try this with the matrix M:

$$M = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$$

a) Use a reflection to transform matrix M into a new matrix N, which does not have a 0 in the upper left corner.

[2]

$$\begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} -3 & 5 \\ 0 & 2 \end{bmatrix}}$$

b) Decompose matrix N from part (a) using the rest of Herreshoff's recipe.

[3]

$$\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -\frac{5}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{5}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\underbrace{\cancel{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}}_{I}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 0 & 2 \end{bmatrix}$$

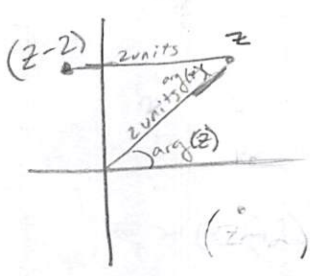
c) Write matrix M as the product of simple shears, stretches, and reflections.

[3]

-1

$$M = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -\frac{5}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

15. Let **z** be a complex number in the first quadrant such that $|z| = 2$. Find $|z-2|$ in terms of arg(**z**). In your work, sketch a picture that includes z and (z - 2). [5]



$|z| =$

$\sqrt{a^2+b^2} = 2$

$z-2 = 2\sin(\arg z) + (2\cos(\arg z) - 2)i$

$2\cos(\arg(z)) - 2$

$|z-2| = \boxed{2\sqrt{2} - 2\cos(\arg z)}$

$|z-2| = \sqrt{4\sin^2 \arg z + 4\cos^2 \arg z + 4 - 8\cos(\arg z)}$

$= \sqrt{4+4-8\cos(\arg z)}$

$= 2\sqrt{2-2\cos(\arg z)}$

-1

16. Consider a prism whose base is an isosceles triangle (but not an equilateral triangle). How many elements would be in the rotation/reflection group of the prism? [3]

$\boxed{4}$

17. Write 3 separate matrices that represents each of the following individual transformations : [6]

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100, 105, 110, 115, 120, 125, 130, 145, 150, 155, 160, 165, 170, 175, 180

a) Rotate 50 degrees counterclockwise

$$\begin{bmatrix} \cos 50^\circ & -\sin 50^\circ \\ \sin 50^\circ & \cos 50^\circ \end{bmatrix}$$

b) Reflect over the line $\theta = 20^0$

$$\begin{bmatrix} \cos 40^\circ & \sin 40^\circ \\ \sin 40^\circ & -\cos 40^\circ \end{bmatrix}$$

c) Reflect over the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

d) How many elements would be in the group generated by your answer from (a) alone? [3]

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e) How many elements would be in the group generated by your answers from (b) and (c) together? [3]

18

20° rotation

18. The matrix $\begin{bmatrix} 3 & 3 \\ 15 & 15 \end{bmatrix}$ maps to the line y = $\underline{5x}$. The set of points along the sides of the unit square, under this transformation, would map to a line segment. Name the two endpoints of the line segment. [5]

$\underline{(0,0)}$ and $\underline{(6,30)}$

$$\begin{bmatrix} 3 & 3 \\ 15 & 15 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 6 & 3 \\ 0 & 15 & 30 & 15 \end{bmatrix}$$

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