Analysis H 2018-19 – Deggeller / Gleason / Hahn $\frac{77}{77}$ Timothy Herchen through Probability Solving

Midterm Exam 1 Period: G

No Calculators on this test. But no reason to simplify your answers either.

Questions 1-4 are Multiple Choice. Circle the best answer. [3 each]

$\binom{3}{0}+\binom{4}{1}+\binom{5}{2}=\binom{6}{2}$

1 4 10 15

1. Which of the following expressions are equivalent to entry $\binom{17}{8}$ in Pascal's Triangle?

I. $\binom{16}{7}+\binom{16}{8}$ II. $\binom{18}{9}-\binom{17}{9}$ III. $\binom{8}{0}+\binom{9}{1}+\binom{10}{2}+\ldots\ldots\binom{16}{8}$

a) I only b) I and II only c) I and III only d) II and III only e) I, II, and III.

$\binom{17}{8}+\binom{17}{9}=\binom{18}{9}$ $\binom{2}{0}+\binom{2}{1}=\binom{3}{1}$ $\binom{5}{2}-\binom{4}{2}=\binom{4}{1}$ 10 - 6 $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$ $\binom{n+1}{k+1}-\binom{n}{k+1}=\binom{n}{k}$

2. $\frac{(n+2)!-n!}{(n+1)!}$ can be factored into a rational function in the form $\frac{ax^2+bx+c}{dx+e}$. Find the sum a + b + c + d + e.

a) 8 b) 3 c) 7 d) 9 e) 5

$\frac{n!\left((n+2)(n+1)-1\right)}{(n+1)!}=\frac{(n+2)(n+1)-1}{n+1}=\frac{n^2+2n+1n+2-1}{n+1}=\frac{n^2+3n+1}{n+1}$

$\rightarrow 4(1)+3=7$

3. As n gets bigger and bigger (goes towards infinity), then the following sum will approach what value?

$1+\frac{1}{2}+\frac{1}{4}+\cdots=2$ $a=1$ $r=\frac{1}{2}$ $\frac{1}{1-\frac{1}{2}}=2$

$$\sum_{k=1}^{n}3\left(\frac{2}{5}\right)^k$$

$\frac{3}{1-\frac{2}{5}}=\frac{3}{\frac{3}{5}}=5$ $5-3=2$

a) 7 b) 7.5 c) 2 d) 5 e) 1.2

$3\cdot\frac{2}{5}+3\cdot\left(\frac{2}{5}\right)^2$

$=3\left(\frac{2}{5}+\left(\frac{2}{5}\right)^2+\cdots\right)$

$\frac{1}{1-\frac{2}{5}}-1$

4. Use telescoping to derive a compact expression for the following sum of even-numbered Fibonacci terms:

$$F_{14}+F_{16}+\ldots\ldots\ldots+F_{200}$$

$=\cancel{F_{15}}-F_{13}+\cancel{F_{17}}-\cancel{F_{15}}+\cdots+F_{201}-\cancel{F_{199}}$

a) $F_{201}-F_{13}$ b) $F_{202}-F_{12}$ c) F_{203} d) F_{202} e) $F_{202}-F_{13}$

6

5. The number 28 can be found in 7 locations in Pascal's Triangle (and the negative Pascal's Triangle). 2 such locations are $\binom{28}{1}$ and $\binom{28}{27}$ but those are boring. Express 28 as 4 **different** binomial coefficients. [4]

$$28 = \binom{8}{2} = \binom{8}{6} = \binom{-3}{6} = \binom{-7}{2}$$

$28 \cdot 2 = 56 \to 7, 8$

$\binom{-2}{7} =$

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-3   1 -3  6 -10 15 -21 28
-2   1 -2  3  -4  5
-1   1 -1  1  -1 -1
 0   1  0  0   0
 1   1  1  0   0
 2   1  2  1   0
 3   1  3  3   1
     1  4  6   4  1
     1  5 10  10  5  1
     1  6 15  15  6  1
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1  6  15  15  6  1
1  7  21  30  21  7  1
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$\binom{-n}{k} = \frac{(-n)!}{(-n-k)!\,k!} = (-n)(-n-1)\cdots$

6. Write the following sum using sigma notation. Then actually calculate the sum (in terms of "m") [5]

$$5+11+17+23+\ldots\ldots\ldots(6m-19)$$

$\sum_{n=4}^{m} 6n-19 \longrightarrow \sum_{n=1}^{m-3} 6(n+3)-19 = \sum_{n=1}^{m-3} 6n-1 = \frac{6(m-3)(m-2)}{2} - (m-3)$

$= 3(m-3)(m-2)-(m-3)$

$= (m-3)(3m-7)$

$= 3m^2-9m-7m+21$

$= 3m^2-16m+21$

Sigma: $\sum_{n=1}^{m-3} 6n-1$ Sum: $3m^2-16m+21$

$m=4 \to -16+21=5$ ✓

$m=5 \to 75-80+21 = 5+11$ ✓

7. Find the coefficient for the $x^{10}y^{25}z^{15}$ term in the expansion of $(3x+2y+z)^{50}$ [4]

coeff of $(3x)^{10}$

ways to arrange 50 factors

$$\boxed{3^{10}\cdot 2^{25}\cdot \frac{50!}{10!\,25!\,15!}}$$

coeff of $(2y)^{25}$ duplicity of x, y, z factors respectively

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$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = \frac{n(n+1)\cdots(n+k-1)(-1)^k}{k!} = \binom{n+k-1}{n-1}(-1)^k$

$n+k-1=8$

$n-1=2$

$n=3$

$k=6$

$n+k-1=8$

$n-1=6$

$n=7$

$k=2$

$\binom{-2}{1} = \binom{2}{1}$ $\binom{-3}{1} = \binom{3}{2}$

0

8. Consider the "triangle of 6's" below. The last row shown is the 4th row. The first term of the nth row can be found by the formula. $F(n) = 3n^2 - 3n + 6$

6 ↙ 1st 1, 5, 1

12 2nd 18 3rd 1 + 4 + 8 + 12 +

24 4th 30 5th 36 6.

42 48 54 60

66 72 78

a) Find an expression for the middle term in the nth row (where n is an odd number). [3]

n^{th} row → $F(n) = 3n^2 - 3n + 6$

Middle term in $6\left(\frac{n-1}{2}\right)$ more than $F(n)$ → $M(n) = 3n^2 - 3n + 6 + 3n - 3 = \boxed{3n^2+3}$

b) Write a compact expression for the product of the first n terms in the triangle above (6)(12)(18)...... using factorials and/or exponents. Your answer will have n in it. [2]

$$\underbrace{(6)(12)(18)\cdots(6n)}_{n \text{ terms}} = \boxed{6^n \cdot n!}$$

9. The method of induction can be used to prove the following statement:

"The expression $a^2 - 1$ is divisible by 8 for all positive odd numbers a"

Properly right out the first three steps in a potential induction proof. **YOU DO NOT NEED TO DO THE ENTIRE PROOF!!!** Please properly label all 3 steps. [5]

1. Prove true for $a=1$:

$(1)^2 - 1 \mid 8$

a | b means a divisible by b

2. Assume true for $a=k$:

$k^2 - 1 \mid 8$

3. Prove true for $a = k+2$:

$(k+2)^2 - 1 \mid 8$

-0

Questions 9-13 are Multiple Choice (Again!?) [3 each]

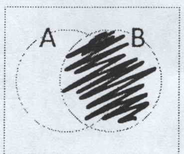
10. How many ways can you split 7 students into 2 groups, where each group has at least one student?

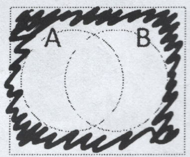
a) 7! b) 128 c) 126 d) 63 e) 64

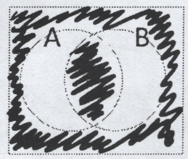
$\frac{2^7}{2} - 1 = 63$

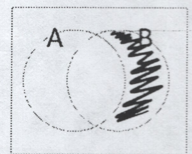
2 choices / student, /2 b.c. duplicates, −1 because can't have all in one group

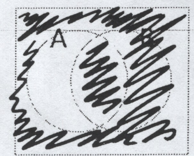
11. Which diagram represents $P(A' \cup B)$?

a) 

b) 

c) 

d) 

e) 

12. Which is logically equivalent to $P(A \cup B')'$?

a) $P(A' \cap B)$ b) $P(A \cap B')$ c) $P(A' \cap B)'$ d) $P(A' \cap B')$ e) $P(A \cap B)$

$(A \cup B')'$

13. How many distinct 3-letter arrangements can you make from the letters in the word "COLTS"?

a) 33 b) 24 c) 60 d) 120 e) 30

${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$

14. How many distinct 3-letter arrangements can you make from the letters in the word "CALLS"?

a) 33 b) 24 c) 60 d) 120 e) 30

Case 1: Only one L → ${}_4P_3 = 4 \cdot 3 \cdot 2 = 24$

Case 2: both L's → $3 \cdot 3 = 9$

More Free Response

15. 7 students randomly arrange themselves into a circle. What is the probability that Ed is standing directly between Edd and Eddy? (obviously assuming that Ed, Edd, and Eddy are 3 of the 7 students) [3]

$\underbrace{\text{Edd Ed Eddy}}_{\text{1 unit}}$ $\overbrace{\cdots}^{\text{4 others}}$ → $\frac{5!}{5} \cdot 2$

$\frac{7!}{7}$

$\rightarrow \dfrac{\frac{5!}{5} \cdot 2}{\frac{7!}{7}} = \dfrac{4! \cdot 2}{6!} = \dfrac{2}{6 \cdot 5} = \boxed{\dfrac{1}{15}}$

16. Jar A contains 2 white and 2 blue marbles. Jar B contains 1 white and 2 blue marbles.

a) A random jar is selected, and then a random marble is taken out of the jar. What is the probability that the marble is blue? [3]

$$\frac{1}{2}\cdot\frac{2}{4}+\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{2}\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{1}{2}\left(\frac{7}{6}\right) = \boxed{\frac{7}{12}}$$

$$\frac{3}{6}+\frac{4}{6}$$

b) A random jar is selected, and then a random marble is taken out of the jar. What is the probability that Jar A was selected, given that the marble is blue? [3]

$$\frac{\frac{2}{4}}{\frac{2}{4}+\frac{2}{3}} = \frac{6}{6+8} = \frac{6}{14} = \boxed{\frac{3}{7}}$$

c) A random marble is selected out of Jar A and placed into Jar B. Then a random marble is selected from Jar B. What is the probability that a blue marble was taken out of Jar A, given that the final marble is blue? [3]

Since P(blue marble taken) = P(white marble taken),

$$P = \frac{\frac{3}{4}}{\frac{3}{4}+\frac{1}{2}} = \frac{\frac{3}{4}}{\frac{5}{4}} = \boxed{\frac{3}{5}}$$

17. I have 2 nickels and 3 quarters in my pocket.

a) If I randomly choose 2 of the coins, what is the probability that I will select one nickel and one quarter? [3]

$$\frac{2\cdot 3}{\binom{5}{2}} = \frac{6}{\frac{5\cdot 4}{2}} = \frac{12}{5\cdot 4} = \boxed{\frac{3}{5}}$$

$$1-\frac{4}{10} = \frac{6}{10}$$

b) If I randomly choose 2 of the coins, what is the expected value of the two coins together? [4]

2 nickels: $\frac{1}{10}$ 2 quarters: $\frac{3}{10}$ nickel + Q: $\frac{6}{10}$

$$E = \frac{1}{10}\cdot 10 + \frac{3}{10}\cdot 50 + \frac{6}{10}\cdot 30 = 1+15+18 = \boxed{34¢}$$

5+5 25+25 5+25

18. In order to gain access to the exclusive We Love Ones Club, you must show your love for 1's by rolling 6 fair, 6-sided dice, and getting at least 2 of dice to show a "1". What is the probability that you will gain access? [4]

$$P(\geq 2 \text{ dice}) = 1 - P(1 \text{ dice}) - P(0 \text{ dice}) = 1 - \binom{6}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5 - \left(\frac{5}{6}\right)^6$$

$$= \boxed{1 - \left(\frac{5}{6}\right)^5 - \left(\frac{5}{6}\right)^6}$$

19. What is the probability of being dealt a 7 card hand in poker (assume a 52 card deck) and getting a Full House (3 of one denomination, 2 of another and 2 "other" cards)? [4]

3 card denom. 3 card to exclude 2 card denom. 2 card include

$52 - 8 = 44$

remaining 2

$$\frac{13 \cdot 4 \cdot 12 \cdot \binom{4}{2} \cdot \overbrace{\left(44 \cdot 43/2 - 11 \cdot \binom{4}{2} \cdot \frac{1}{2}\right)}^{\text{remaining 2}}}{\binom{52}{7}}$$

For remaining 2 cards, we subtract $11 \cdot \binom{4}{2} \cdot \frac{1}{2}$ to undo overcounting when the remaining 2 cards have the same denomination (11 denoms left, $\binom{4}{2}$ to choose 2 cards, $\frac{1}{2}$ to count them at $\frac{1}{2}$ weight since they will be counted twice)

1 - 0