Analysis H – Deggeller / Gleason / Hahn

GAtM Exam 2019 – [80 points]

NO CALCULATOR

75/80

A Geometric Approach to Matrices Timothy Hentler

Period: 6

1. Consider the three groups:

Group A: Hexagonal Prism under Rotation

Group B: Hexagon under rotation

Group C: The 6-post snap group

For each group below, write the letter (A, B, or C) of its isomorphic group. If the given group is not isomorphic to any of the above groups, write "X". [2 pts each]

a) B is isomorphic to the group generated by $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ under multiplication: $\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$

b) A is isomorphic to the dihedral (rotation/reflection) group D_6

c) A is isomorphic to the group generated by a hexagonal pyramid under reflection

d) A is isomorphic to the group generated by a hexagon under reflection

e) X is isomorphic to the group generated by a hexagonal prism under reflection

f) B is isomorphic to the group generated by the multiplication group of $\left\{cis \frac{k\pi}{3}\right\}$, where k is an integer.

Incorrectly written

2. [3 pts] **Multiple Choice (circle the best answer):** In the graph shown to the right, the preimage points A and B were transformed into A' and B' through which simple transformations?

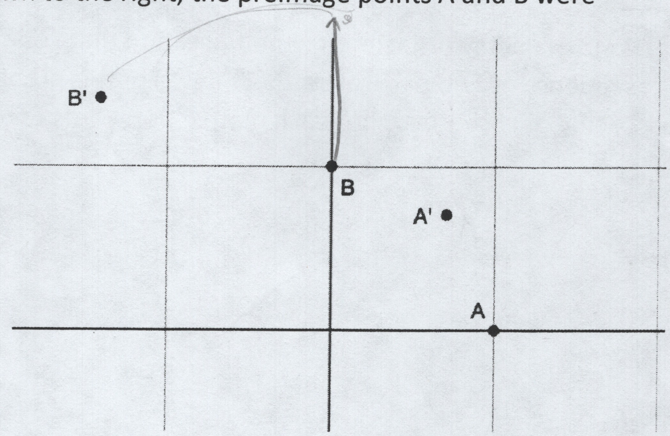
a) dilation followed by rotation

b) stretch followed by rotation

c) shear followed by rotation

d) rotation followed by a shear

e) shear followed by a reflection



3. [3 pts] **Multiple Choice (circle the best answer):** In the Herreshoff method of matrix decomposition, we can take any invertible 2x2 transformation matrix and express it in terms of

a) stretches/reflections and dilations

b) rotations and dilations

c) dilations and shears

d) mapping to line and mapping to point

e) shears and stretches/reflections

-0

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$-1\ I$

4. [3 pts] **Multiple Choice (circle the best answer):** The two matrices given above represent transformations of the points in 3D coordinate space. Taken together, they generate a group that is isomorphic to ______________.

ok

a) triangle under reflection   b) square under reflection   c) cube under reflection

d) tetrahedron under reflection   e) cube under rotation

A AB

5. [5 pts] **For this question, circle ALL correct answers.** The set of real numbers is the same size as:

a) the set of complex numbers   b) the set of rational numbers   c) the integers

d) the set of points in the plane   e) the set of two by two matrices.

6. [5 pts] **For this question, circle ALL correct answers.** The matrices $\begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix}$ and $\begin{bmatrix} \cos(-1) & -\sin(-1) \\ \sin(-1) & \cos(-1) \end{bmatrix}$ (both in radians) generate a group under multiplication which is isomorphic to which group(s)?

a) rotation of the 360-gon   b) integer powers of 3 under multiplication

c) the complex numbers under multiplication   d) the integers under addition

e) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ under multiplication

4573

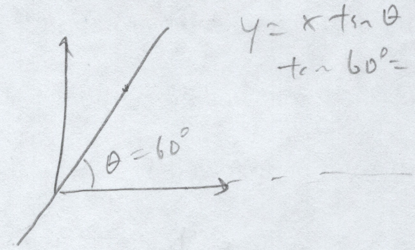
biggest botfly (Bolivian)

7. [5 pts] Show using matrix multiplication that you can produce a counterclockwise rotation of 120 degrees by a sequence of two particular reflections. For this problem, each element in each matrix should be a number (not in terms of sine or cosine). Under each matrix, use words to describe what that particular matrix does (be specific).

insect

$y = x \tan\theta$

$\tan 60° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$



$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

not 120° ccw

Descriptions: reflect over $\theta = 60°$ reflect over x-axis

/-3

8. [4 pts] Write a single 3x3 matrix that could be used to map the plane to the line $y = \frac{5}{2}x + 4$

$$\boxed{\begin{bmatrix} 2 & 2 & 0 \\ 5 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 2x+2y \\ 5x+5y+4 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} 5(x+y) \\ 2(x+y)+4 \end{matrix}$$

$5a = x$ $\quad a = \frac{x}{5}$

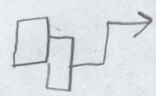
$2a + 4 = y$

9. [4 pts] Name two sets that have a countably infinite number of elements. Then clearly show and/or describe a one-to-one correspondence for the two sets, to show that they are the same size.

Set 1: $\mathbb{N}$ (natural number) Set 2: $2\mathbb{N}$ (even natural numbers)

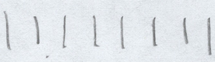
One-to-One correspondence:

$$\mathbb{N} \ni x \longleftrightarrow 2x \in 2\mathbb{N}$$

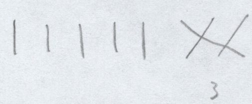


10. [5 pts] Draw a specific element of the 8-post snap group that has (or state "not possible")...

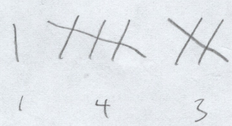
a) Period of 1.



b) Period of 3.



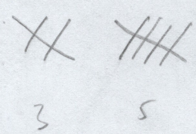
c) Period of 12.



d) Period of 14.

not possible

e) Period of 15.



chair

11. [3 pts] Wally thinks that the 4-post snap group is isomorphic to the reflection group of a square. Write a few sentences to Wally explaining why he's wrong. No need to be lengthy here - just give enough evidence to disprove Wally's misconception.

the four post snap group has $4! = 24$ elements, while the reflection group of a square D_4 has 8 elements. Since the orders are different, they must be not isomorphic.

-0

15. [6

$\begin{bmatrix} 2 & -5 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} -29 \end{bmatrix}$

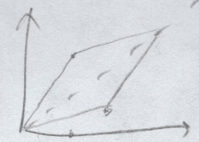
12. Consider the matrix $A = \begin{bmatrix} 3 & 5 \\ 7 & 2 \end{bmatrix}$

a) [2 pts] Find detA.

det $A = \boxed{-29}$

BIG MATRIX

b) [3 pts] Your answer from part (a) relates to the area of a parallelogram ABCD. What are the coordinates of the vertices of the parallelogram, and what is its area?



Vertices: (0,0), (3,7), (5,2), (8,9) Area: 29

c) [3 pts] Areas can't be negative, but your answer to part (a) is. Explain this discrepancy. In what cases would the determinant give you a "negative area"?

It is because of the order the vertices appear. If the matrix was rewritten $\begin{bmatrix} 5 & 3 \\ 2 & 7 \end{bmatrix}$, the det would be +29. The determinant gives a negative area, as a geometric interpretation, when the first column is the bottom right point relative to the parallelogram's diagonal with the origin.

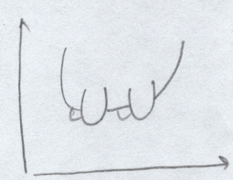
13. [4 pts] Consider a kite (K) with vertices at (-3,0) (3,0) (0,2) and (0,-5). If you were to transform K using the transformation matrix $\begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}$, you would get a line segment. Find the endpoints of the line segment.

-1

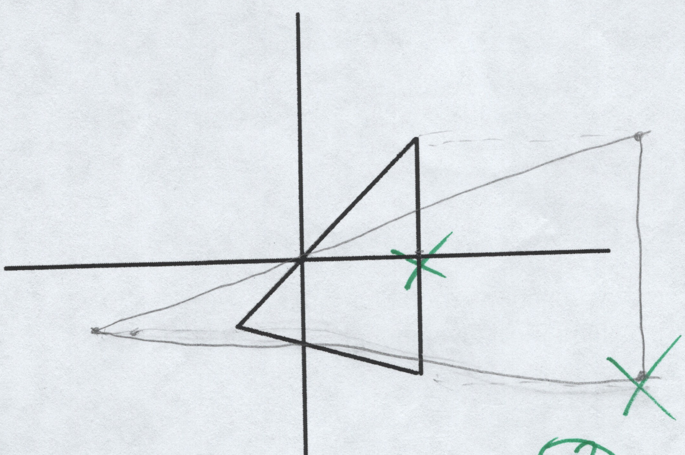
$\boxed{(-15,-60) \text{ and } (6,24)}$

$\begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}\begin{bmatrix} -3 & 3 & 0 & 0 \\ 0 & 0 & 2 & -5 \end{bmatrix}$

$= \begin{bmatrix} -3 & 3 & 6 & -15 \\ -12 & 12 & 24 & -60 \end{bmatrix}$



14. [4 pts] On the same axis, graph the image of the triangle below under an x shear of 3. Be as accurate as possible



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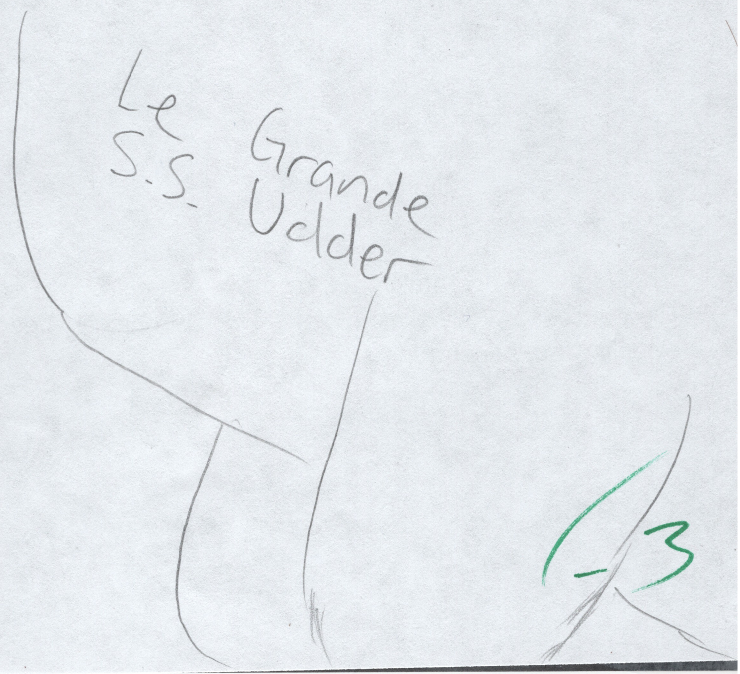

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15. [6 pts] Use the decomposition method taught in class to explain what the matrix $M = \begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix}$ does.

Clearly show your work and report your answer using words.

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix}}_{M}\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\overset{M}{\begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix}}$$

$$\underset{M^{-1}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 4 & 4 \end{bmatrix}$$

$-4 \cdot \frac{2}{3} + 4$
$= 4(-\frac{2}{3} + 1) = \frac{4}{3}$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix}$$

M does the following, in order:

-2

1 stretch in x by 3 2 sheer in y by 4 3 stretch in y by $\frac{4}{3}$ 4 sheer in x in $\frac{2}{3}$

16. [6 pts] The circle below is a unit circle. On the coordinate axis, graph (and label) the complex numbers a, b, and c such that the given information holds true.

Given:

$|a| = 1.2$ ✓

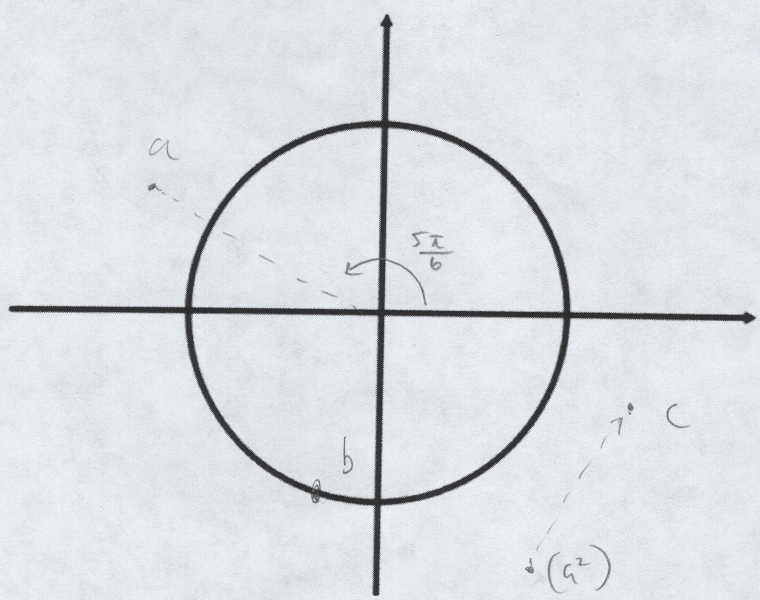
$Arg(a) = \frac{5\pi}{6}$

$Re(b) < Im(a)$

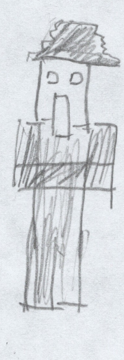
$Im(b) < 0$

$|b| = 1$

$c = a^2 - b$



lake



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