boxes of

9/27/19

Analysis Midterm 1 19-20 Alan Lee is independent of grade D

Deggeller/Gleason/Tantod

ATPS [36 pts]

1. Express all of the following as a compact expression of as few terms as possible [3 pts each]:

a) $F_{20} + F_{21} + F_{22} + \cdots + F_{90} + F_{91}$ $\quad$ $F_{93} - F_{21}$

$\cancel{F_{21}} - F_{19} + \cancel{F_{22}} - F_{20} + \cancel{F_{23}} - \cancel{F_{21}} \cdots F_{91} - \cancel{F_{89}} + F_{92} - \cancel{F_{90}}$

$F_{93} - F_{21}$

b) $\binom{77}{1} + \binom{77}{3} + \binom{77}{5} + \cdots \binom{77}{75} + \binom{77}{77}$ $\quad$ 2^{76}

$2^{77}/2$

c) $\sum_{k=5}^{\infty} \frac{3}{7^k}$ $\quad$ $\frac{1}{4802}$

$\frac{3}{7^5} + \frac{3}{7^6} + \frac{3}{7^7} \cdots = \frac{3}{7}\left(1 + \frac{1}{7} + \frac{1}{49} \cdots\right) = \frac{3}{7^4}\left(\frac{7}{6 \cdot 2}\right) = \frac{1}{2 \cdot 7^4}$

d) $\frac{\binom{n}{n-2}}{\binom{n-2}{n-4}}$ (assume n is a whole number bigger than 4). $\quad$ $\boxed{n(n-1)/(n-2)(n-3)}$

$\frac{n!}{(n-2)!\cancel{2!}} \Big/ \frac{(n-2)!}{(n-4)!\cancel{(2)!}} \qquad \frac{n!(n-4)!}{(n-2)!(n-2)!} = \frac{n(n-1)}{(n-2)(n-3)}$

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e) $\prod_{n=1}^{k}(2n)$ $\quad$ $\boxed{2^k \cdot k!}$

$2 \cdot 4 \cdot 6 \cdots 2k = 2^k(1 \cdot 2 \cdot 3 \cdots k) = 2^k \cdot k!$

2. The sum of all the multiples of 2 from 2 to 1,700 = 723,350

The sum of all the multiples of 17 from 17 to 1,700 = 85,850

a) In one sentence, explain why the sum of all the multiples of 2 and 17 between 2 and 1,700 is not 723,350 + 85,850. [2 pt] We are double counting the numbers that are both multiples of 2 AND 17 if we add 723,350 to 85,850

b) Find the sum of all the multiples of 2 and 17 between 2 and 1,700 (no need to simplify) [3 pts]

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$34 + 68 + 102 \cdots 1700 = 34(1 + 2 + \cdots 50) = \frac{34 \cdot 50 \cdot 51}{2} = 43{,}350$

$723{,}350 + 85{,}850 - 43{,}350 = \boxed{765{,}850}$

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3. Pierre thinks he has a formula for the sum of the first n cubes: $\frac{(n(n+1))^2}{4}$ where n is a whole number.

Show at least two non-zero examples of specific positive whole numbers n where Pierre is right.
[2]

$\sum$ first 1 cubes = 1 $\qquad \frac{(1(1+1))^2}{4} = \frac{(2)^2}{4} = 1$ ✓

$\sum$ first 3 cubes = 1+8+27=36 $\qquad \frac{(3(3+1))^2}{4} = \frac{144}{4} = 36$ ✓

Now that you've established a couple base cases, use induction to show that Pierre is indeed correct. Please be proper about your proof so your steps are obvious. [6]

① Base case for n=1 the sum of the first 1 cubes $= 1 = \frac{(1(1+1))^2}{4}$ ✓

② Assume true for n=k $\qquad 1^3+2^3\dots k^3 = \frac{(k(k+1))^2}{4}$

③ Prove true for n=k+1, in other words prove $1^3+2^3\dots(k+1)^3 \stackrel{?}{=} \frac{((k+1)(k+2))^2}{4}$

$$1^3+2^3\dots(k+1)^3 = (1^3+2^3\dots k^3)+(k+1)^3 = \frac{(k(k+1))^2}{4} + \frac{4(k+1)^2(k+1)}{4}$$

$$= \frac{(k+1)^2(k^2+4(k+1))}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{((k+1)(k+2))^2}{4}$$ QED by mathematical induction.

4. If you were to expand the trinomial $(2a - 3b + c)^{100}$ you'd have a lot of terms. One of them would have an $a^{10}b^{70}c^{20}$ in it. What would the coefficient of this term be? [4]

$$(2a)^{10}(-3b)^{70}(c)^{20}\binom{100}{10}\binom{90}{70}\binom{20}{20}$$

$$\Rightarrow \boxed{2^{10}\cdot 3^{70}\cdot\frac{100!}{10!\,70!\,20!}} \qquad \frac{100!}{\cancel{90!}\,10!}\cdot\frac{\cancel{90!}}{70!\,20!}$$ ✓

5. Consider the "even" triangle below (first 4 rows given).

		2		→ 1·2
	4	6		→ 2·3
8	10	12		→ 3·4
14	16	18	20	→ 4·5

Find the last term of the 53rd row. Show the work that leads to your answer. [4]

last term of nth row = n(n+1)

last term of 53rd row = $53(53+1) = \boxed{2862}$

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Probability Section: [36 points]

1. A fair 6-sided die is being rolled twice. What is the probability that the same face (#) comes up twice in a row? [2]

1,1 4,4
2,2 5,5
3,3 6,6

$\Rightarrow \frac{6 \text{ good}}{6^2 \text{ total}} = \boxed{\frac{1}{6}}$

2. I write the eleven letters in "Mississippi" in random order. What is the probability that the last letter is an i? [2]

4 i's, 11 total

$\boxed{\frac{4}{11}}$

3. Erica wrote the five letters of her name in random order. What is the probability that they are all in the right place? [2]

$\frac{1}{5!} = \boxed{\frac{1}{120}}$

4. I pick six cards at random from a standard deck of 52 cards.
a) What is the probability that there are exactly three hearts and two clubs in my hand? [3]

$\boxed{\frac{\binom{13}{3}\binom{13}{2}\binom{26}{1}}{\binom{52}{6}}}$

b) What is the probability I have three of a kind (three of one denomination and three other cards that don't match). [3]

$\boxed{\frac{\binom{13}{1}\binom{4}{3}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{6}}}$

c) I look at my hand and notice there are 4 hearts. What is the probability one of them is an ace? [2]

$\boxed{\frac{4}{13}}$

$\frac{\cancel{12} \cdot \cancel{11} \cdot \cancel{10} \cdot \cancel{4} \cdot \cancel{3} \cdot 2}{13 \cdot \cancel{12} \cdot \cancel{11} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{10}}$ 2

d) I look at my hand and Have 3 Aces and 3 non-aces. The dealer offers me an opportunity to buy one more card for $10. I know that if I end up with 4 Aces the casino will pay me $1000. Should I buy the additional card? Justify your answer mathematically. [3]

There are 46 cards left in the deck and 4-3=1 of them are aces, The probability of getting an ace next is therefore $\frac{1}{46}$. The expected value of your winnings is then $\frac{1}{46} \cdot 1000 - 10 = \frac{540}{46} \approx \$11.74$. Since the EV is positive, you should buy the additional card

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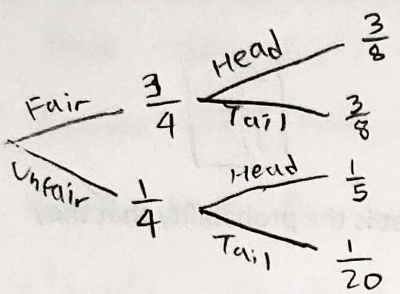
5. a) I have an "unfair" coin that turns up heads 80% of the time. I flip it 10 times. What is the probability that I get exactly 8 heads? [2]

$$\boxed{\binom{10}{8}(0.8)^{8}(0.2)^{2}} \approx 30.2\%$$

b) I mix the "unfair" coin in with three other normal "fair" coins and shuffle them around. I pick a random coin out of the four and start flipping it. I flip a head on the first flip. What is the probability that I picked the unfair coin? [3] Show the work that leads to your answer.



$$\frac{\frac{1}{5}}{3/8+1/5} = \frac{\frac{8}{40}}{15/40+8/40} = \boxed{\frac{8}{23}}$$

6. 40% of Gunn students earned an A last semester in Math. 30% earned an A in Science.
a) If these results are independent of each other, what percent of the students earned an A both in science and in math? [2]

$$(0.4)(0.3) = \boxed{12\%}$$

b) what percent of students didn't get an A in either class? [2]

$1-0.12-0.18-0.28$
0.42

$$(1-0.4)(1-0.3) = \boxed{42\%}$$

$1-(0.4+0.3-0.12)$
$= 1-(0.58) = 0.42$

c) What is the probability that they got an A in science *given that they received an A in math*? [2]

	Math	No
Sci	12%	18%
No	28%	42%

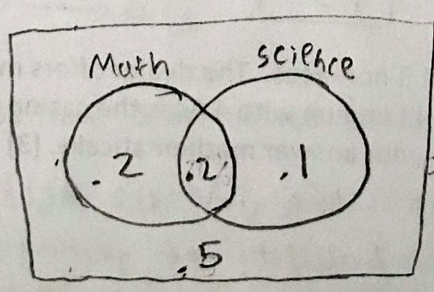
they are independent.

$$\frac{12}{12+28} = \frac{12}{40} = \frac{3}{10}$$

$\boxed{30\%}$

For part d assume that the grades ARE NOT independent any more. Assume 20% earned an A in both classes now.

d) What is the probability if I pick a student at random that they earned an A in at least one of the courses? [2]



$$\frac{.5}{1} \Rightarrow \boxed{\frac{1}{2}}$$

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Awesome!

7. At Burger IM you choose one of 4 buns, and one of 5 cheeses (or no cheese). Then you can add as many (or as few) toppings as you want. A sign in their store says that there are 40 million different burgers possible. How many different toppings do they have? Show the work that leads to your answer [3].

n toppings

$$\underset{\text{buns}}{4} \cdot \underset{\text{cheeses}}{6} \cdot 2^n \approx 40,000,000$$

$\log_2 1666666 \approx 20.668$ $\quad 2^n \approx 1,666,666$ $\quad \boxed{\text{21 toppings}}$

8. Suppose you are tracing a path along the coordinate plane moving from (0,0) to (9,9) by moving only right and up one unit at a time. If your path is random, what is the probability that you will pass through the point (5, 5)? [3]

18 moves, 9 rights

$$\binom{18}{9}$$

10 moves 5 rights $\times$ 8 moves 4 rights

$$\binom{10}{5} \quad \binom{8}{4}$$

$$\boxed{\frac{\binom{10}{5}\binom{8}{4}}{\binom{18}{9}}}$$