Analysis Honors 2021-2022 – Hahn / Tantod
Geometric Approach to Matrices Exam
NO CALCULATORS

69.5
74 points

Chris Lee, shear talent
Period: ____

1. [4 pts] Write the 4 characteristics of a group:

closure ___ associativity ___ invertability ___ Identity ___

2. [4 pts] Is each of the following groups isomorphic to D_4, the dihedral group of a square? Write "yes" or "no".

a) The 4-post snap group? no

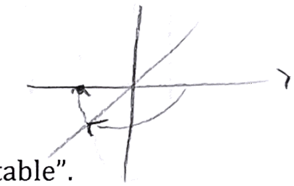
b) The rotation group of a tetrahedron? no

c) The rotation group of a square-based prism? yes

rotate abt z reflect x-y plane

d) The group generated by $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$? yes

$x \to -y$
$y \to x$



3. [7 pts] For each set and operation named below:
   - If it is not a group, write "N/A"
   - If it is a finite group, write the order of the group.
   - If it is an infinite group, write either "C" for "countable" or "UNC" for "uncountable".

a) real numbers under addition UNC

b) integers under multiplication N/A

c) square under reflection 8

d) 3D space under the transformation $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ 2

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) triangular prism under rotation 6

f) 5-posts under the snap operation 5!

g) positive and negative odd numbers under addition N/A

4. [4] Consider a regular hexagon drawn on the complex axes with center at the origin and a vertex at $1 + 2i$.

a) Find $|1 + 2i|$.

$\sqrt{5}$

b) Use your answer from part (a) to find the hexagon's perimeter.

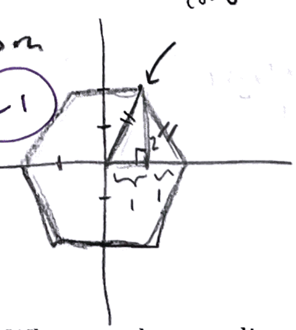
· regular hexagon so the angled sides will all be the same

congruent bc SAS

· we know top and bottom sides = 2 $2 \cdot 2 + 4 \cdot \sqrt{5} =$

-1

$\boxed{\text{perimeter} = 4 + 4\sqrt{5}}$



5. Consider the matrix $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$

a) [1] Find det A

$6 - 20 = -14$

$\boxed{\det = -14}$

b) [3] Your answer from part (a) relates to the area of a parallelogram ABCD. What are the coordinates of all 4 the vertices of the parallelogram, and what is its area?

Vertices: (0,0), (3,4), (8,6), (5,2)   Area: 14

-1

For Questions 6-9, all matrix elements should be given in simplified, exact form (not in terms of trig functions).

6. [6] Write a simplified 2x2 matrix that will accomplish each transformation

a) Rotate by $\theta = \frac{\pi}{3}$

$$\begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}$$

b) Reflect over the line $y = x\tan\frac{\pi}{2}$

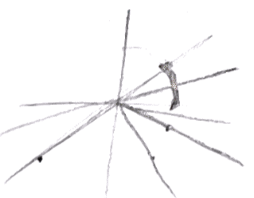
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$\boxed{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

c) Shear in y by a factor of 2.

$$\boxed{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}$$

7. [4] Use matrix multiplication to show that you can produce a counterclockwise rotation of 60 degrees by a sequence of two reflections. Under each matrix, use words to describe what that matrix does (be specific).



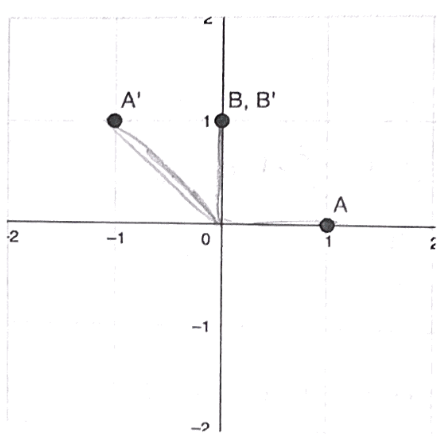
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \longrightarrow \text{CCW rotation } 60°!$$

reflect over x-axis   reflect over line $\theta = -30°$

8. [4] Write a single 3 x 3 matrix that could be used to map the plane to the line $y = \frac{5}{2}x + 4$

$$\boxed{\begin{bmatrix} 2 & 2 & 0 \\ 5 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}}$$

9. [4] In the graph below, the preimage points A and B were transformed into A' and B' through what sequence of transformations? Give your answer in BOTH forms: as a multiplication of 2x2 matrices (don't actually multiply them), and as a written description of the transformations in sequence.



Matrix Multiplication:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ← identity/unit square

Written description:

shear in y by factor of 1 then reflect over the y-axis

$$\begin{array}{cc} A & B \end{array} \qquad \begin{array}{cc} A & B \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

-0

10. [4] Tom Brady is trying to decompose $T = \begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$ into a series of simple stretches and shears. Part of his work is shown below:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{12} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Use Tom Brady's work to fill in the blanks below, expressing T as a series of simple stretches and shears:

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

11. [4] The complex number $z = \frac{(-2+2i)^8}{(\sqrt{3}-i)^3}$ can be simplified to $a + bi$ form where a and b are integers. Find a and b.

$\frac{24\pi}{4} = 6\pi$

$(-2+2i)^8 = \left(2\sqrt{2}\text{ cis } \frac{3\pi}{4}\right)^8 = 4096 \text{ cis } 0$

$\theta = \frac{3\pi}{4}$

$r = 2\sqrt{2}$

$(\sqrt{3} - i)^3 = \left(2 \text{ cis } -\frac{\pi}{6}\right)^3 = 8 \text{ cis } -\frac{\pi}{2}$

$\theta = -\frac{\pi}{6}$

$r = 2$

$\sqrt{3}$, 1, 2

$\frac{4096 \text{ cis } 0}{8 \text{ cis } -\frac{\pi}{2}} = 512 \text{ cis}\left(\frac{\pi}{2}\right)$

$512 \cos\frac{\pi}{2} + 512 i \sin\frac{\pi}{2}$

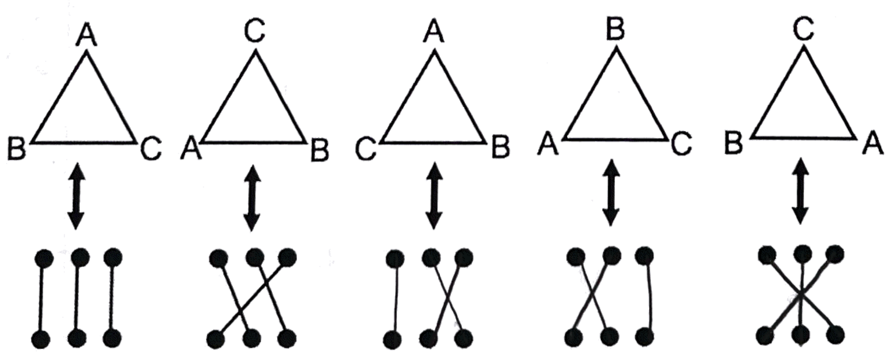
$0 + 512i$

$a = 0$

$b = 512$

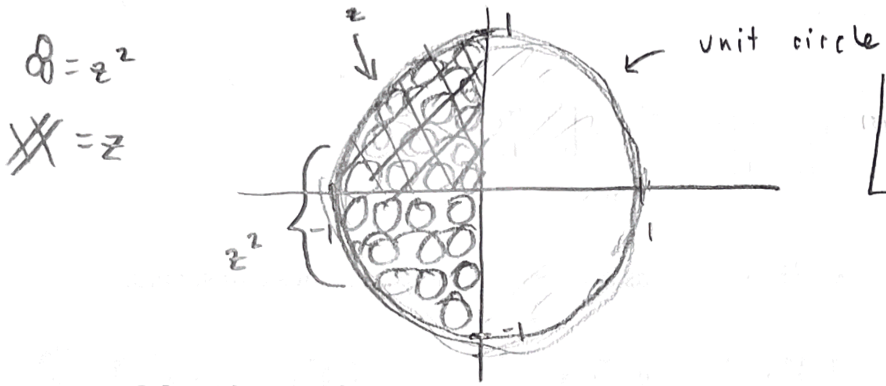
$4096 \div 8 = 512$

12. [3] The dihedral group of a triangle (D3) is isomorphic to the 3-post snap group, a 1-to-1 correspondence can be drawn between their elements. Some of the paired elements are drawn below (and the first two are already completed). Draw in the missing 3 remaining 3-post snap group elements.



-6

13. [4] Given: $|z| = 1$, $Re(z) \le 0$, and $Re(z^2) \le 0$. What are all the possible values (in radians) of $Arg(z)$? In your answer, include a drawing that shows possible positions of z and z^2.



$$\boxed{\frac{\pi}{2} \le \arg z \le \pi}$$

-1.5

14. [8] On the complex axes on the right, the circle has a radius of 1, and complex numbers A, B, and C are shown.

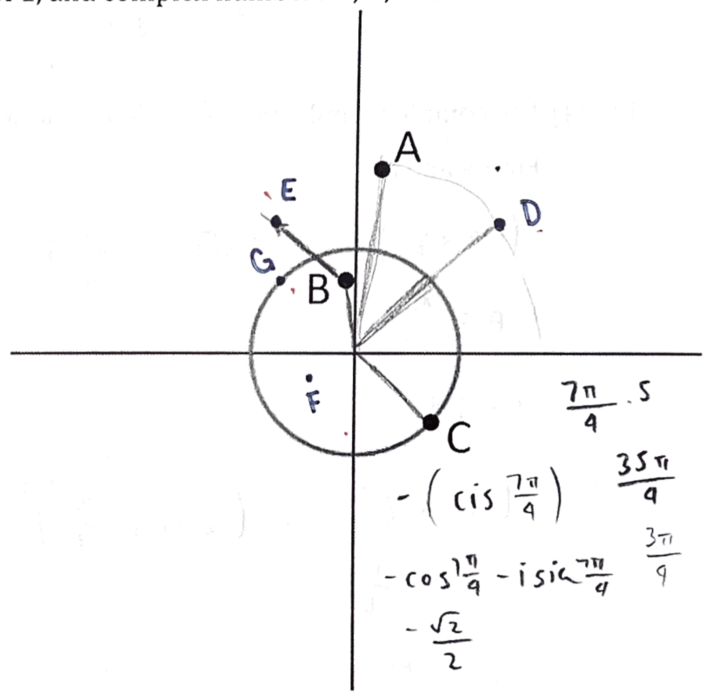
Plot points D, E, F, and G on the same axes:

D = AC

E = B – C

F = B^2

G = C^5



$-\left(\text{cis}\,\frac{7\pi}{4}\right)$

$-\cos\frac{7\pi}{4} - i\sin\frac{7\pi}{4}$

$-\frac{\sqrt{2}}{2}$

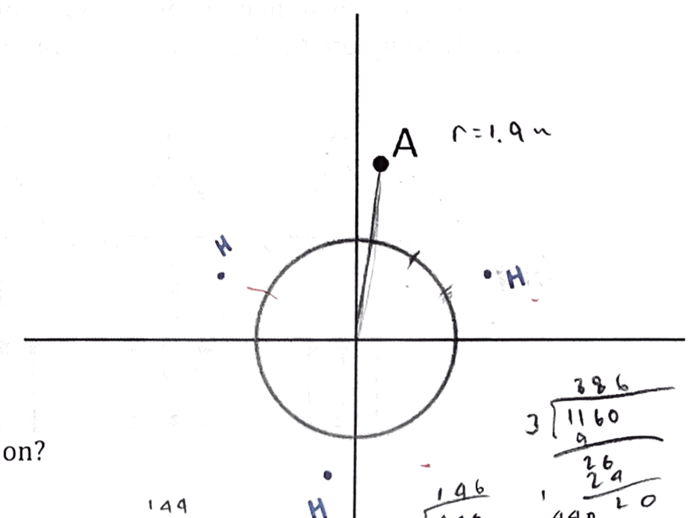
$\frac{7\pi}{4} \cdot 5$

$\frac{35\pi}{4}$

$\frac{3\pi}{4}$

15. On the complex axes on the right, the circle has a radius of 1, and complex number A is shown.

a) [4] If $H^3 = A$, graph all possible values of H.

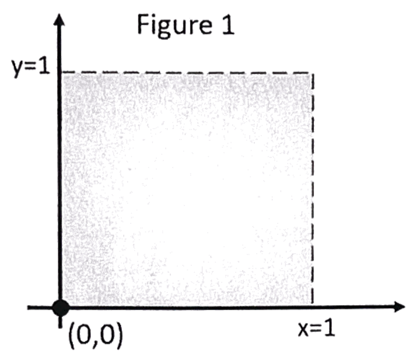


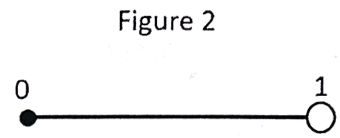
b) [2] Do the values of H form a group under multiplication? Justify your answer (short justification!).

Yes, they form a group of C_3 since they're equally spaced and have the same radius.

2

16. [4] Andy and Beth are trying to compare the number of points on the figures below:

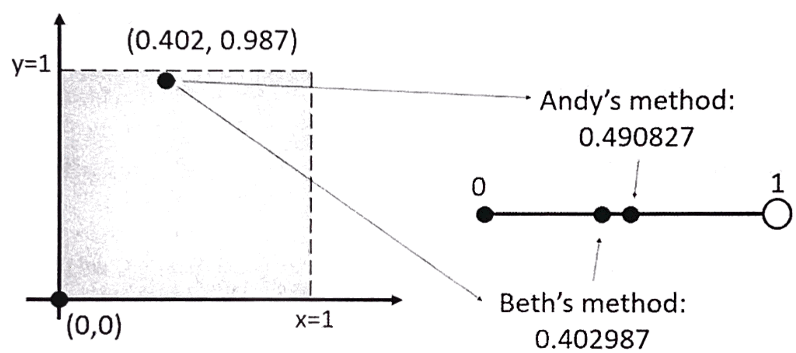




They agree that any point on either figure can be written in decimal form, but they don't agree on the best way to make a correspondence from the coordinate pair in the 1st figure to the single number in the 2nd. They use the point (0.402, 0.987) as an example.

Andy's method: You can alternate between the x- and y-digits: (0.402 , 0.987) → 0.490827

Beth's method: You can write the x-digits, followed by the y-digits: (0.402 , 0.987) → 0.402987



Whose method is will NOT work for all real points from the first diagram? Use a counterexample to prove why that person's method will not work for any point from the first figure.

Beth's method will not work. This is because if you had the point (0.4029, 0.87) it would have the same decimal as (0.402, 0.987) under her method. Since we know that pts on line = pts on plane, this method doesn't allow us to have a 1-1 correspondence. Beth's method also wouldn't work for point's like (0.123, 0.956) vs (0.1234, 0.56) and (0.456, 0.789) vs (0.45, 0.6789)

-0