Analysis H 2022-2023 – Hlasek / Hahn / Tantod
Midterm Exam 1 – AtPS and Probability
NO CALCULATORS

87 / 88 pts 99/100

Midterminator: Justin Oh
Period: 7

Part I: Algebra through Problem Solving

Multiple Choice. Circle the best answer. [3 pts each]

1. Consider a set of 10 numbers, where all of the numbers are equal to 20. Let A = "the arithmetic mean of the 10 numbers" and G = "the geometric mean of the 10 numbers". Which of the following statements is true?

a) A = G = 20
b) A = G ≠ 20
c) A > G
d) ~~G < A~~

$\sqrt[10]{20^{10}}$

2. Which expression is equivalent to $\binom{-3}{50}$?

a) $-\frac{(50)(51)(52)}{2}$ b) $2(51)(52)$ c) $-\frac{51}{2}$ d) $-\frac{6}{4!}$ e) $\frac{(51)(52)}{2}$

$\frac{(-3)(-4)\cdots(-52)}{50!} = \frac{51\cdot 52}{2} =$

3. $F_{15} + F_{17} + F_{19} + F_{21} + \cdots + F_{405} =$

a) $F_{404} - F_{15}$ b) $F_{407} - F_{16}$ c) $F_{404} - F_{14}$ d) $F_{406} - F_{15}$ e) $F_{406} - F_{14}$

$F_{16} - F_{14} + F_{18} - F_{16} + \cdots + F_{406} - F_{404}$

$F_{16} - F_{14} + F_{404} - F_{402}$
$+ F_{406} - F_{16} + F_{406} - F_{404}$
$+ F_{20} - F_{14}$

4. Consider the numbers arranged into a triangle on the right. The last term of the 2nd row is 6, and the last term of the 3rd row is 12. What is the last term of the 100th row?

a) 9999 b) 9900 c) 10302

d) 10100 e) 10200

$100(101) = 10100$

Triangular arrangement:

2

4 6

8 10 12

14 16 18 20

22 24 26 28 30

FD

AtPS Free Response

5. Expand and simplify the sigma expression below. Your answer should be a quadratic expression in the form $ax^2 + bx + c$. [4 points]

$$\sum_{k=0}^{n}(10 - 4k)$$

$$\frac{(10+10-4n)(n+1)}{2} = (10-2n)(n+1)$$

$$= \boxed{-2n^2+8n+10}$$

6. Given the infinite series: $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \frac{32}{81} + \cdots =$

a) Express the series in Sigma notation. [4 points]

$$\sum_{i=0}^{\infty}\left[2\left(-\frac{2}{3}\right)^i\right]$$

b) Find the sum of the series. [4 points]

$$\frac{2\left(1-\left(-\frac{2}{3}\right)^{\infty}\right)}{1+\frac{2}{3}} = \frac{2}{5/3} = \boxed{\frac{6}{5}}$$

7. Simplify completely: $\binom{n}{n-2} + \binom{n+1}{n-1}$ [5 points]

$$\frac{n!}{(n-2)!\cdot 2!} + \frac{(n+1)n!}{(n-1)!\cdot 2!}$$

$$= \frac{n!}{(n-2)!\cdot 2!} + \frac{(n+1)n!}{(n-1)!(n-2)!\cdot 2!} = \frac{n!\left[(n-1)+(n+1)\right]}{2!\cdot(n-1)!} = \frac{n!(2n)}{2\cdot(n-1)!} = \boxed{n^2}$$

-0

8. If you were to expand and completely simplify $\left(x-\frac{2}{x}\right)^{48}$, one of the terms would be of the form Ax^{14}. Find A. Your answer can be left in choose/exponential notation. [4 points]

$a-b=14$
$a+b=48$
$2a=62$
$a=31$
$b=17$

$A=\binom{48}{31}(-2)^{17}$

$\boxed{A=-\binom{48}{31}\cdot 2^{17}}$

△ 9. If $1+5+9+13+\cdots+a_n=780$, solve for n. [5 points]

$4(n-1)+1=4n-3$

$\frac{1+(4n-3)}{2}\cdot n=780$

$\frac{4n-2}{2}\cdot n=780$

$(2n-1)n=780$

$2n^2-n-780=0$

$n=\frac{1}{4}+\sqrt{\left(\frac{1}{4}\right)^2+\frac{780}{2}}$

$=\frac{1}{4}+$

$2n^2-n=780$

$(20^2)2-20=780$

$780\cdot 2=1400+160=1560$

$\boxed{n=20}$

10. Prove by mathematical induction: 3^n+7^n-2 is a multiple of 8. [6 points]

Assume 3^k+7^k-2 is a multiple of 8

Prove $3^{k+2}+7^{k+2}-2$ is a multiple of 8

$=9\cdot 3^k+49\cdot 7^k-2$

$=(3^k+7^k-2)+8\cdot 3^k+48\cdot 7^k$

$=(3^k+7^k-2)+8(3^k+6\cdot 7^k)$

multiple of 8 by assumption; multiple of 8

Base cases: $n=1,2$

$3^1+7^1-2=8$ ✓

$3^2+7^2-2=9+49-2=56=8\cdot 7$ ✓

□.

-0

Part 2: Probability

Questions 1-7 are Multiple Choice. Circle the best answer. [3 points each]

1. Out of a regular deck of cards, you draw a single card. Let A = "the card is a Jack" and B = "the card is a heart". We can say that events A and B are...

~~a) Independent~~

b) Independent, but not Mutually Exclusive

c) Mutually exclusive, but not Independent

d) Both Independent and Mutually Exclusive

e) Neither Independent nor Mutually Exclusive

2. How many ways can you make distinct rearrangements of the word "TIKTOK"?

a) 720 b) 360 c) 180 d) 120 e) 40

$\frac{6!}{2! \cdot 2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2} = 30 \cdot 6 = 180$

3. In "5-card Stud" Poker, you draw 5 cards (without replacement) out of a regular deck.

The expression $\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$ is equivalent to the probability of drawing a...

a) Full House (3 of any one value, and 2 of another value)

b) 3 of a kind (3 of any one value, with the other 2 cards not matching)

c) Full House of Jacks and Queens (3 Jacks and 2 Queens)

d) 3 of a kind Jacks (3 Jacks and 2 other cards not matching)

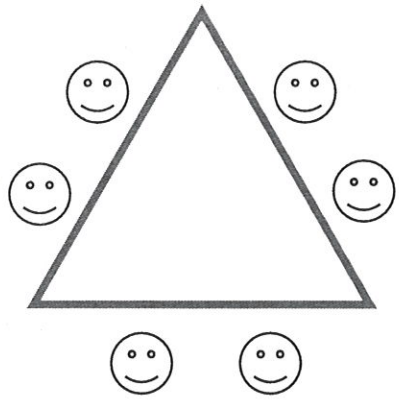
e) none of the above

4. You're getting a smoothie at Jamba Juice, and they ask if you'd like to "boost" your smoothie with extra ingredients. They offer 6 different "boosts" (chocolate, berries, wheatgrass, protein powder, bee pollen, and grasshoppers) but you cannot take more than 2, and you cannot order the same boost twice (you also have the option of taking only 1 boost, or taking no boosts at all). How many different options do you have?

a) 22 b) 21 c) 30 d) 36 e) 37

$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} = 1 + 6 + 15 = 22$

Questions 5 and 6 refer to the equilateral triangle table shown on the right, with 6 seats around the table. People at the table only care about their positions relative to the table and each other, not to the room around the table.



5. How many ways can 6 people be arranged around the table?

a) 720 b) 120 c) 360 (d) 240 e) 1920

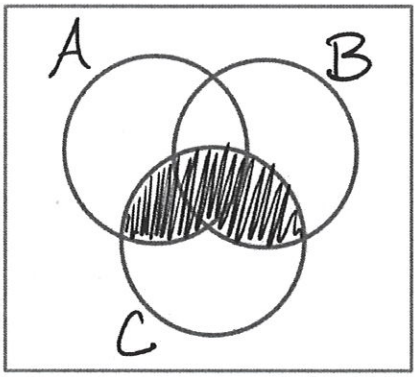
$\frac{6!}{3} = 6 \cdot 5 \cdot 4 \cdot 2 = 30 \cdot 8 = 240$

6. How many ways can 6 people be arranged around the table if Andy and Beth need to share a side?

a) 16 (b) 48 c) 120 d) 240 e) 80

$\frac{3 \cdot 4!}{3} \cdot 2 = 4 \cdot 3 \cdot 4 = 48$

7. Which expression best describes the shaded region of the Venn Diagram on the right?



a) $(A \cup B) \cup C$ (b) $(A \cup B) \cap C$

c) $A \cup (B \cap C)$ d) $A \cap (B \cup C)$

e) $A \cap (B \cap C)$

8. Which of these math questions represent Binomial (Bernoulli) Trials? **(CIRCLE ALL THAT APPLY) [3 points]**

a) You flip 50 coins and remove any that are heads. Then you flip the remaining coins and remove any that are heads. You do this a total of 5 times. What is the probability that you will have removed at least 40 coins?

(b) You play the Powerball lottery once a week for 50 weeks, and record whether or not you're able to match at least one number each time. What is the probability that you will match at least one number on exactly 5 of the weeks?

(c) You have a teacher that randomizes the seating chart every class period. What is the probability that you will sit next to your best friend exactly 4 times in 20 class periods?

d) You go out to the Gunn parking lot and record how many cars are in the lot each hour of the day, starting at 7AM and ending at 7PM. What is the probability that there will be at least 40 cars in the lot?

e) You try to make a PRIME appointment with your math teacher every Wednesday. Sometimes you get in, and others you don't, because the PRIME is full. What is the probability that you were able to get into the PRIME exactly 6 times out of 15 consecutive attempts?

F-0

Probability Free Response.

9. A certain Analysis H class has 6 freshmen, 9 sophomores, and 15 juniors. [each question is worth 2 points]

a) If I select 3 random students from the class, what is the probability that they will all be juniors?

30

$$\frac{\binom{15}{3}}{\binom{30}{3}}$$

b) If I select 3 random students from the class, what is the probability that I will select 1 freshman, 1 sophomore, and 1 junior?

$$\frac{6 \cdot 9 \cdot 15}{\binom{30}{3}}$$

c) If I select 3 random students every day for 100 days, what would you expect to be the average number of juniors selected each day?

$$\frac{\binom{15}{3}}{\binom{30}{3}} \cdot 3 + \frac{\binom{15}{2}\binom{15}{1}}{\binom{30}{3}} \cdot 2 + \frac{15 \cdot \binom{15}{2}}{\binom{30}{3}} \cdot 1$$

d) If I randomly award a gold, silver, and bronze medal to three students in the class (one medal per student, and the same student cannot win twice), what is the probability that a junior will receive the gold medal, a sophomore will receive the silver medal, AND a freshman will receive the bronze medal?

$$3 \cdot \frac{6 \cdot 9 \cdot 15}{30 \cdot 29 \cdot 28} = \frac{27}{28 \cdot 29}$$

-1 e) If I randomly arrange all the students into one big circle, what is the probability that all the freshmen are sitting next to each other?

$$\frac{24! \; 15! \cdot 6!}{29!}$$

f) For 100 days, I give a random student a cookie every day (the same student can win multiple times). What is the probability that at least 2 of the cookies are given to freshmen? Give your answer as a compact expression, but no need to simplify. $\frac{24}{30} = \frac{4}{5}, \frac{6}{30} = \frac{1}{5}$

$$1 - \left(\frac{4}{5}\right)^{100} - \binom{100}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{99}$$

F 1

10. In the same Analysis H class as problem #9 (6 freshmen, 9 sophomores, and 15 juniors), one student is picked at random. If the student is a freshman, then a 6-sided die (numbered 1-6) is rolled. If the student is a sophomore or junior, then a 10-sided die (numbered 1-10) is rolled. Evaluate each probability: [2 points each]

a) P(roll a 4) = $\boxed{\frac{17}{150}}$

$\frac{4}{5} \cdot \frac{1}{10} + \frac{1}{30} = \frac{4}{50} + \frac{1}{30}$

$= \frac{12}{150} + \frac{5}{150}$

b) P(roll a 4 | the student is a freshman) = $\boxed{\frac{1}{6}}$

c) P(student is a freshman | roll a 9) = $\boxed{0}$

d) P(student is a freshman | roll a 4) = $\boxed{\frac{5}{17}}$

$\frac{1/30}{17/150} = \frac{1}{30} \cdot \frac{150}{17}$ 5

e) P(The math teachers are running out of creative ideas | the last 10 math questions were about the same Analysis class) =

infinite number of test questions, infinite number of 10 questions about some identical math class

$= \frac{(\frac{\infty}{\infty})}{(\frac{\infty}{\infty})}$ = indeterminate ☺

-0