Analysis H – Hahn/Hlasek/Tantod
GAtM Exam 2023 [74 points]
NO CALCULATOR

76.5 / 77 points
99.4%

A Geometric Approach to: Justin Oh
Period: 7

For this whole exam, we're working in 2D, unless 3D is specified in the question.

Multiple Choice [3 pts each]

1. Given z in the third quadrant,

   a) $Re(z) < 0$ b) $Re(z) > 0$ c) $Re(z) = 0$ d) Not enough info

2. Given z in the third quadrant,

   a) $Im(z^2) < 0$ b) $Im(z^2) > 0$ c) $Im\,(z^2) = 0$ d) Not enough info

3. The rotation group of a triangular prism contains how many elements?

   a) 6 b) 12 c) 24 d) 48

4. Which matrix will take a pre-image, shear in the y-direction by 3, and then rotate by 30 degrees?

   a) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{3\sqrt{3}+1}{2} & \frac{3-\sqrt{3}}{2} \end{bmatrix}$ b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{3\sqrt{3}+1}{2} & \frac{-3+\sqrt{3}}{2} \end{bmatrix}$ c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{3\sqrt{3}-1}{2} \\ \frac{1}{2} & \frac{3+\sqrt{3}}{2} \end{bmatrix}$ d) $\begin{bmatrix} \frac{\sqrt{3}-3}{2} & -\frac{1}{2} \\ \frac{1+3\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

   $$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+3}{2} & -1/2 \\ \frac{1+3\sqrt{3}}{2} & \sqrt{3}/2 \end{bmatrix}$$

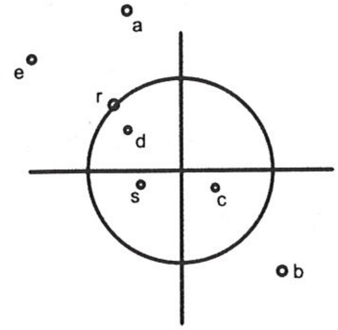
5. Working in the 3D system, which matrix will take a pre-image and rotate it 90 degrees around the y-axis?

   $$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

   a) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

6. Given complex numbers r and s below and the given unit circle, which complex number most closely represents rs?

   a) a
   b) b
   c) c
   d) d
   e) e



Multiple Choice (continued):

7. The set of all points in a line is the same size as to the set of (circle ALL that apply)…

   a) points in a plane b) integers c) rational numbers

   d) real numbers e) points on a line segment

For Questions 8 and 9, refer to the following matrices:

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \text{rot. of } \frac{\pi}{3} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad AB = [$$

8. What is the order of the group generated by matrix A?

   a) 2 b) 4 c) 6 d) 12 e) 24

9. Matrix AB is a

   a) Rotation by 60 degrees
   b) reflection over the line y = xtan120
   c) reflection over the line y =xtan30
   d)
   d) rotation by 30 degrees
   e) reflection over the line y = xtan60

For Questions 10-12, refer to the matrices $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **and** $D = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

10. If we just use matrix C as a generator, we make ____________

    a) not a group b) a group with order 2 c) a group with order 4

    d) a countably infinite group e) an uncountably infinite group

11. If we just use matrix D as a generator, we make ____________

    a) not a group b) a group with order 2 c) a group with order 4

    d) a countably infinite group e) an uncountably infinite group

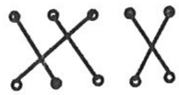
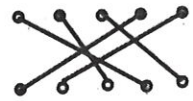
12. If we use both matrices C and D as generators, we make ____________

    a) not a group b) a group with order 4 c) a group with order 8

    d) a countably infinite group e) an uncountably infinite group

Free Response

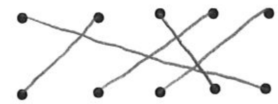
1. Below, are two beautiful members of the 5 post group [2 each]

Q:  R: 

a) State the period of Q 6

△ b) State the period of R 6

c) Draw Q o R (Q snap R where R goes first) to the right:



6 d) Draw the inverse of R

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

120

e) How many elements total are there in the five post snap group? $5! = 120$

f) Would element R generate the entire 5 post group? Explain how you know.
no, R has a period of 6, so it can only generate 6 elements before returning to itself.

△• 2. Show using matrix multiplication that you can produce a counterclockwise rotation of 120 degrees by a sequence of two particular reflections. For this problem, each element in each matrix should be a number (not in terms of sine or cosine). Under each matrix, use words to describe what that particular matrix does (be specific).[3]

$$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

Descriptions: reflection across $y = x\tan 60°$; reflection across x-axis

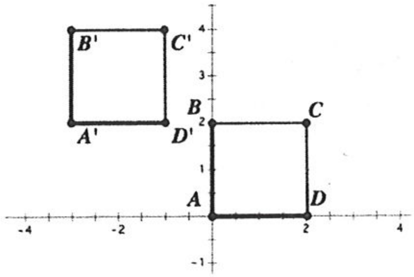
3. [4 pts] Write a single 3x3 matrix (yes, a 3 x 3!) that could be used to map the plane to the line $y = \frac{2}{3}x - 2$

$$T = \begin{bmatrix} 3 & 3 & 0 \\ 2 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

-0

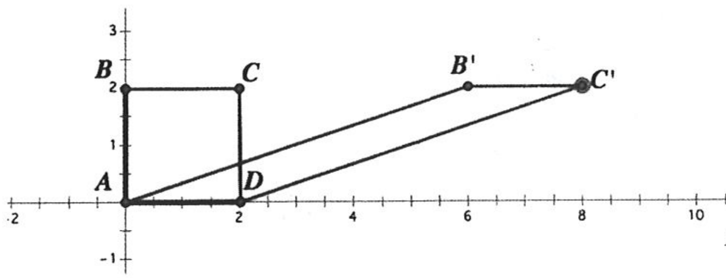
4. Write a single matrix that would turn the pre-image square in each figure shown below (its side lengths are 2) into the corresponding image. [2 pts each]

a)



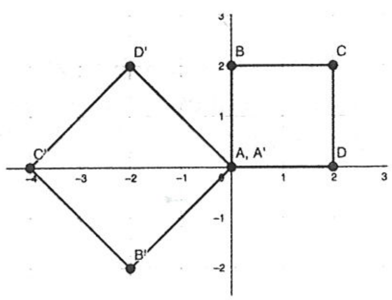
Answer for (a): $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

b)



Answer for (b): $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

c)



Answer for (c): $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$

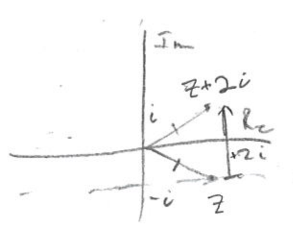
5. If $(z + 2i)^{15} = z^{15}$, then Im(z) = ? Explain with a diagram and a sentence.(3 pts)

$|(z+2i)^{15}| = |z^{15}|$

$|(z+2i)| = |z|$

$\boxed{Im(z) = -1}$

Since the magnitudes of z and (z+2i) must be equal, Im(z) must equal -1 since that is the only way adding 2i keeps the magnitude the same.



$z = a+bi$

$\left(\sqrt{a^2+b^2} = \sqrt{a^2+(b+2)^2}\right)^2$

$\cancel{a^2}+b^2 = \cancel{a^2}+(b+2)^2$

$\pm b = b+2$

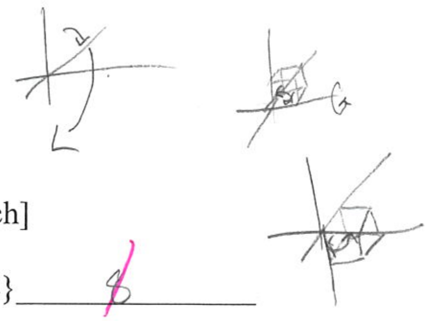
$2b = -2$

$b = -1 \quad \Rightarrow Im(z) = -1$

-0

6. Working in 3D, consider the following transformation matrices

$$D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



a) How many elements are in the group generated by...[2 each]

-0.5 {D} 4. {E} 2. {D and E} ~~8~~

$$ED = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad DE = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

7. Working in 3D, given $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) What transformation does F do? [2]

stretch by 5 in y

b) Would F alone generate a group? Explain why or why not [2]

No, F alone is unable to generate inverses. Inverse of a stretch by 5 is a stretch by 1/5, However, multiplying F alone only generates stretches by positive, integer powers of 5, ($1/5 = 5^{-1}$).

8. As we learned first semester, we can write the equation of a parabola parametrically as (t, t^2).

Use matrix transformations to find the parametric equation of the same parabola after a rotation of 30 degrees counterclockwise. [3]

x = $\dfrac{-t^2 + t\sqrt{3}}{2}$ y = $\dfrac{t^2\sqrt{3} + t}{2}$

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} t \\ t^2 \end{bmatrix} = \begin{bmatrix} \frac{t\sqrt{3}}{2} - \frac{t^2}{2} \\ \frac{t}{2} + \frac{t^2\sqrt{3}}{2} \end{bmatrix}$$

$\boxed{-0}$