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Analysis Honors 23/24
Hahn / Hlasek / Tantod
Unit 8 Biggish Quiz: Sequences and Series
No calculators.
Trigonometric functions have arguments in radians.

Serial converger: Grace Gao
Period: 7

42 44.8
50 pts

Part I: Sequences (20 pts)

$n^0 \to$ always $1 \to$ converges
$n^1 \to$ diverges
$\hookrightarrow$ 1 is not included

Question #1-3 are multiple choice. Circle one best answer. [1pt each]

1. Sequence $\{n^p\}$ converges if and only if

   a) $p \geq 0$ b) $p \geq 1$ c) $p \leq 0$ d) $p < -1$ or $p \geq 1$ e) $-1 < p \leq 1$

   ✓

2. Which of the sequences below is everywhere increasing and bounded above? (convergent)

   a) $\{\sin n\}$ b) $\left\{\sin\left(\frac{\pi n}{2}\right)\right\}$ c) $\{n^2 - 3n + 2\}$ d) $\left\{-\frac{1}{2^n}\right\}$ e) $\left\{\frac{n}{n-1}\right\}$

   ✓ $\hookrightarrow -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}$

   always increasing, approaching 0

   decreasing

   $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$

3. ~~$\left\{\frac{7}{r^{2n}}\right\}$ converges if and only if~~

   a) $r > 0$ b) $r < 0$ c) $r \geq 1$ d) $r > 1$ e) $r \leq 1$ f) $r < 1$

4. Answer True or False for each statement below. [1 pt each]

   a) If a sequence converges, it must have an upper bound. true ✓

   b) If $a_n < b_n$ for all natural n and $\{b_n\}$ converges, then $\{a_n\}$, must converge. false ✓

   b_n

   a_n

   c) If a sequence $\{a_n\}$ is divergent and satisfies $a_n \geq 0$ for all natural n, it must increase without bound. false ✓

   still has an upper bound

5. Find the limit of each sequence or state that it diverges. No justification needed. [2 pts each]

   a) $\left\{\frac{3n}{\sqrt{4n^2+1}}\right\}$

   $\frac{3}{\sqrt{4}} = \boxed{\frac{3}{2}}$ ✓

   b) $\{a_n\}$, where $a_n = 5\sqrt{a_{n-1}}$ and $a_1 = 5$

   $a_1 = 5$

   $a_2 = 5\sqrt{5}$

   $a_3 = 5\sqrt{5\sqrt{5}}$

   $a_4 = 5\sqrt{5\sqrt{5\sqrt{5}}}$

   $\boxed{\text{diverges}}$

   c) $\left\{\frac{n^n}{n!}\right\}$

n	a_n
1	-1
2	$\frac{4}{2} = 2$
3	$\frac{27}{6}$
4	$\frac{256}{24}$

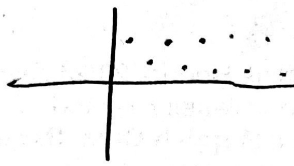
   numerator grows significantly faster than denominator

   $\downarrow$

   $\boxed{\text{diverges}}$ ✓

-2

ways to show divergence:



-2

6. Consider the sequence $\{b_n\}$, where $b_n = \begin{cases} 3 & \text{if n is odd} \\ \frac{6}{2n+1} & \text{if n is even} \end{cases}$.

State the value to which the sequence converges, or state that the sequence diverges. CLEARLY justify your answer. If your work is correct but is difficult to interpret, you may not receive full credit. [4 pts]

The **value** to which the sequence converges (or say "diverges"): diverges

Show your work here:

when n is odd, the sequence converges to 3

But when n is even, the sequence converges to 0

n	b_n
1	$\frac{6}{3}=2$
2	$\frac{6}{5}$
3	$\frac{6}{7}$
4	$\frac{6}{9}$
5	$\frac{6}{11}$
100	$\frac{6}{201}$

always decreasing

Because $\{\frac{6}{2n+1}\}$ is always in the neighborhood for $n \geq M$ when $M = \frac{6-\varepsilon}{2\varepsilon}$, $\{\frac{6}{2n+1}\}$ converges to 0. But because the odd numbers converge to 3 while the even numbers converge to 0, the overall sequence diverges.

justify using ∞ terms outside either neighborhood.

Neighborhood proof:

$-\varepsilon \leq \frac{6}{2n+1} \leq \varepsilon$

$-2n\varepsilon - \varepsilon \leq 6 \leq 2n\varepsilon + \varepsilon$

$-2n\varepsilon \leq 6+\varepsilon$ $\qquad$ $6-\varepsilon \leq 2n\varepsilon$

$n \geq -\frac{6+\varepsilon}{2\varepsilon}$ $\qquad$ $n \geq \frac{6-\varepsilon}{2\varepsilon}$

always true

7. For the sequence $\left\{\frac{6^n}{(2n-1)!}\right\}$, state the value to which the sequence converges, or state that the sequence diverges. CLEARLY justify your answer with a proof. For your proof, you MUST use one of the following tests: the squeeze theorem OR the big theorem (bounded above/below and always increasing/decreasing theorem) OR the ~~comparison principle.~~ If your work is correct but is difficult to interpret, you may not receive full credit. [4 pts]

$5! = 1\cdot2\cdot3\cdot4\cdot5$

24

24×42 $\quad$ 48 $\quad$ 96 $\quad$ 1008

36 $\quad$ 6 $\quad$ 216 $\quad$ 6 $\quad$ 1296

Test/Principle used: big theorem

The **value** to which the sequence converges (or say "diverges"): 0

Show your work here:

n	a_n
1	$\frac{6}{1}=6$
2	$\frac{36}{6}=6$
3	$\frac{216}{120}$
4	$\frac{1296}{1008}$
5	$\frac{6\cdot6\cdot6\cdot6\cdot6}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}$

36

$\frac{72}{7}$ $\quad$ 504

216

20×504

$(n+1)(n!) = (n+1)!$

$\frac{6^n}{(2n-1)!} \geq \frac{6^{n+1}}{(2(n+1)-1)!}$

$n=1$ $\quad$ $4\cdot3!$

$6^n(2n+1)! \geq (2n-1)!\,6^{n+1}$

$\cancel{6^n}(2n+1)(2n)\cancel{(2n-1)!} \geq \cancel{(2n-1)!}\cancel{6^n}\cdot6$

$4n^2+2n \geq 6$

$2n^2+n-3 \geq 0$

$2n^2+3n-2n-3 \geq 0$

$2n(n-1)+3(n-1) \geq 0$

$(2n+3)(n-1) \geq 0$

always true ; always decreasing

$\frac{6^n}{(2n-1)!} \geq 0$

$6^n \geq 0$

always true

0 is a lower bound

✓

-2

sum of $a_n = 1$.

$\sum_{n=1}^{\infty} a_n = 1$ $\frac{1}{2}$

$\sum_{k=1}^{n} a_k$

Part II: Series (30 pts)

8. Consider a sequence $\{a_n\}$ and its corresponding partial sums $s_n = \sum_{k=1}^{n} a_k$. Given that $\sum_{n=1}^{\infty} a_n = 1$, determine the following limits. No explanation required. [1 pt each]

   a) $\lim_{n\to\infty} a_n =$ $\underline{0}$ ✓ b) $\lim_{n\to\infty} s_n =$ $\underline{1}$ ✓

9. Consider a sequence $\{b_n\}$ and its corresponding series $\sum_{n=1}^{\infty} b_n$. For each statement below, give an example that satisfies the given requirements, or state that none exists. [2pts each]

   ☆ a) $\{b_n\}$ is everywhere increasing and $\sum_{n=1}^{\infty} b_n$ converges. $b_n =$ $\underline{-\frac{1}{n^2}}$ ✓

   $-\frac{1}{n^2}$

   b) $\{b_n\}$ alternates and $\sum_{n=1}^{\infty} b_n$ diverges. $b_n =$ $\underline{(-1)^n \cdot 7}$ ✓

   $\frac{3}{4}$, $\frac{9}{16}$

   $\frac{a_1}{1-r}$ $\frac{1}{1-r} = 4$

   ☆ c) $\{b_n\}$ converges and $\sum_{n=1}^{\infty} b_n$ converges to 4. $b_n =$ $\underline{\left(\frac{3}{4}\right)^{n-1}}$ ✓ $\frac{1}{1-\frac{3}{4}} = \frac{1}{1/4} = 4$

10. For each series below, write "C" if it converges and "D" if it diverges. No justification needed. [2 pts each]

    $\ln n < n$ sequence doesn't converge to 0

    a) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ $\frac{1}{n}$ diverges b) $\sum_{n=1}^{\infty} \frac{2n^2 + 3n + 1}{\sqrt{9n^4 + n + 7}}$ c) $\sum_{n=1}^{\infty} \frac{3^n + 1}{2^{2n}}$

    $\underline{D}$ ✓ $\underline{D}$ ✓ $\underline{C}$ ✓

$$\frac{3^n \cdot 3 + 1}{2^{2(n+1)}} \cdot \frac{2^{2n}}{3^n + 1} =$$

$$\frac{((3^n \cdot 3) + 1) \cdot \cancel{2^{2n}}}{(\cancel{2^{2n}} \cdot 4) \cdot (3^n + 1)} =$$

$$\frac{\cancel{3}^n \cdot 3 + 1}{4 \cdot \cancel{3}^n + 4}$$ ratio < 1

10

11. For each series below, write a clear proof to show the convergence or divergence of the series. Indicate if the series converges or diverges and name the test used. If you choose to use more than one test for a series, state names of all the tests used. [5 pts each]
Important: for the two series below, you MAY NOT use the same test twice!
(If you use the same test for both series, you will lose 3 points.)

a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{(n+1)!}$

Circle one: converges / diverges

Test used: alternating series test

① Alternating signs ✓ ✓

$(-3)^1 = -$

$(-3)^2 = +$

$(-3)^3 = -$

...

factorials always positive

③ $\lim \frac{(-3)^n}{(n+1)!} \to 0$ ✓ ✓

Numerator is multiplied ×3 each term. Denominator is always multiplied by 3 or greater each term. Therefore, the denominator increases significantly more and the series converges to 0.

② Abs. value always decreasing ✓

$$\left|\frac{(-3)^n}{(n+1)!}\right| \geq \left|\frac{(-3)^{n+1}}{(n+2)!}\right|$$

$$\frac{3^n}{(n+1)!} \geq \frac{3^n \cdot 3}{(n+2)!}$$

$\cancel{3^n}(n+2)\cancel{(n+1)}! \geq \cancel{3^n} \cdot 3\cancel{(n+1)}!$

$n+2 \geq 3$

$n \geq 1$ ✓

always true!

-4

b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2\sqrt{n}}$

$\ln(n) > 0$

$\ln(n) < n$

Circle one: converges / diverges

Test used: Comparison principle

$$0 < \frac{1}{(n)^{5/2}} \leq \frac{\ln n}{n^2\sqrt{n}}$$

converges due to p-series test.

$p \geq 1$

b. A. B.

C. B. A.

$\cancel{n^2\sqrt{n}} \leq \ln(n)\cancel{(n)^{5/2}}$

$1 \leq \ln(n)$

always true

$\therefore \sum \frac{\ln}{n^2\sqrt{n}}$ converges

You proved that the given ~~sequence~~ series is greater or equal to something convergent.
That does not provide any insight into convergence of the given series.

4

12. Consider the region $\mathcal{R}$ bounded by the function $f(x) = \frac{x^3}{3}$ and the x-axis in the interval [0,3] as shown on Diagram 1 below.

Diagram 1: region $\mathcal{R}$

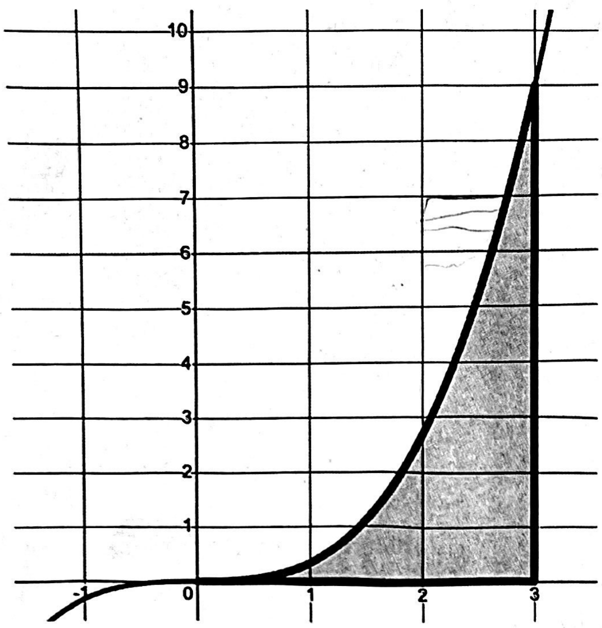
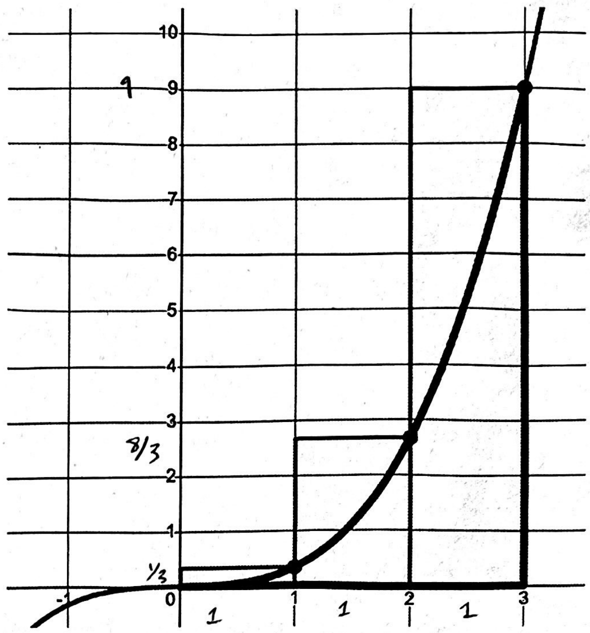


Diagram 2: area A_1



a) Find the area A_1 of the region shaded in Diagram 2 that consists of three rectangles with width 1 whose upper right corners lie on the graph of $y = f(x)$. [1 pt]

$\frac{1}{3} + \frac{8}{3} + 9 = \boxed{12}$ ✓

b) Circle one: Area of the region $\mathcal{R}$ is smaller ✓ greater than the area A_1. [1 pt]

c) Find an algebraic expression in terms of n for the area A_n of the region that consists of $3n$ rectangles with width $\frac{1}{n}$ whose upper right corners lie on the graph of $y = f(x)$. [3 pts]

Potentially helpful formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\frac{1}{n}\left[\left(\frac{(\frac{1}{n})^3}{3}\right)+\left(\frac{(\frac{2}{n})^3}{3}\right)+\dots\left(\frac{(\frac{3n}{n})^3}{3}\right)\right]$$

$$\frac{1}{n}\left[\frac{1}{3}\left(\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\dots\left(\frac{3n}{n}\right)^3\right)\right]$$

$$\frac{1}{n}\left[\frac{1}{3}\left(\frac{\left(\frac{3n(3n+1)}{2}\right)^2}{n^3}\right)\right]$$

$$\frac{1}{n}\left[\frac{1}{3}\left(\frac{(9n^2+3n)^2}{4n^3}\right)\right]$$

$$\boxed{A_n = \frac{81n^4+54n^3+9n^2}{12n^4}}$$ ✓

d) Determine the area of region $\mathcal{R}$ by finding $\lim_{n\to\infty} A_n$. [1pt]

$\frac{81}{12} = \frac{27}{4} = \boxed{6.75}$ ✓

$\frac{21}{4}$

$3\overline{)81}$, 27, 2, 54

12, 7, 84

$3\overline{)81}$, 27, 6, 21

5.2

$4\overline{)21}$, 20, 10, 8

6.75

$4\overline{)27}$, 24, 30