Analysis H - Hahn / Hlasek / Tantod
GAtM Exam, 2023-2024
No Calculators

91.6
71
80 points

Groupie: Michi Tanaka
Period: 6

1. For a certain complex number z, Re(z)>0 and Im(z)>0. Answer "Always", "Sometimes" or "Never" for each of the following statements. [1 pts each]

a) $Re(z^2) > 0$ some times ✓

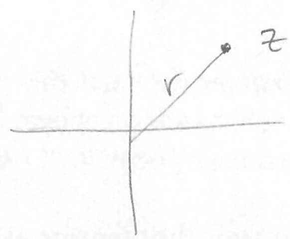
b) $Im(z^2) > 0$ always ✓

c) $Re(z - 2i) > 0$ always ✓

d) $Im(z - 2i) > 0$ sometimes ✓

e) $Re(\bar{z}) > 0$ always ✓

f) $Im(\bar{z}) > 0$ never ✓



$z^2 = r^2 \text{cis } 2\theta$

$\bar{z} = a - bi$

2. For each set listed below, give the number of elements in the set, or write "C" if the set is countably infinite, or "UNC" if the set is uncountably infinite. [2 pts each]

a) Points on a line segment UNC ✓

b) Rational Numbers excluding 0 C ✓

c) Prime Numbers C ✓

-2 d) Elements in the Cantor Set C ✗

-2 e) Elements in the rotation/reflection group of a cube 30 ✗

90° rotation
$4 \times 3 = 12$
$3 \times 6 = 18$ + = 30

f) Elements in the 6-post snap group 720 ✓

$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

g) Elements in the 4-post snap group that have a period equal to 2 9 ✓

4! elements total 24

h) Elements in the group generated by "rotate 5 degrees" and "rotate -5 degrees" 72 ✓

i) Elements in the group generated by "rotate 5 radians" and "rotate -5 radians" C ✓

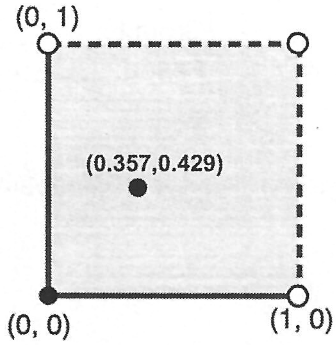
j) Complex numbers in the form a + bi, where a and b are integers C ✓

$z = a + bi = \sqrt{a^2+b^2} \text{ cis } \tan^{-1}\left(\frac{b}{a}\right)$

$360/5 = 72$

Same as plane, every point (integer) can be mapped to a number line.

-4



3. One way to map the points within a 2D square (let's call this "Set A") onto the points on a line segment ("Set B") is to first create a coordinate system for each, as shown above. Then take the (x,y) coordinate of a point in SET A and **alternate the digits** to create a corresponding point in SET B.

a) Fill in the table of values to complete the mapping as described above. The first is done for you. [2 pts each]

SET A point	**SET B point**
(0.357, 0.429)	0.345279
(0.123, 0.456)	0.142536 ✓
(0.45, 0.78) ✓	0.4758
(0.3, 0.987)	0.390807 ✓

b) Andy doesn't like the mapping method used in the table in part (a), and would rather just add the x and y-values of the Set A point to get a Set B value: (0.357, 0.429) → 0.786. Explain, using at least one counterexample, why this would **not** be a one-to-one correspondence between Set A and Set B. [3 pts]

If we take point (0.6, 0.6) which exists inside set A, there won't be a corresponding point 1.2 in set B since it is out of bounds. ✓

c) Beth also doesn't like the mapping method used in the table in part (a), and wants to map the points from Set A to Set B this way: (0.357, 0.429) → 0.357429. Explain, using at least one counterexample, why this would **not** be a one-to-one correspondence between Set A and Set B. [3 pts]

If we have point (0.9, 0.99) that would map to 0.999, but another DIFFERENT point (0.99, 0.9) will map to the same 0.999 in set B. 2 points in set A maps to the same point in set B, so it will not be 1 to 1. ✓

-0

4. Is there a one-to-one correspondence between the set of complex numbers and the set of 2x2 transformation matrices? If yes, describe the one-to-one correspondence and include a non-zero example. If no, give an example of an element in one set that does not have a corresponding element in the other set. [3 pts] −2

Yes $z = r\text{cis}\,\theta$ maps to $\begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$ and each value in the complex set has a unique correspondence in the 2x2 matrices.

Example: $z = \text{cis}\,30° \leftrightarrow \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

What about $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

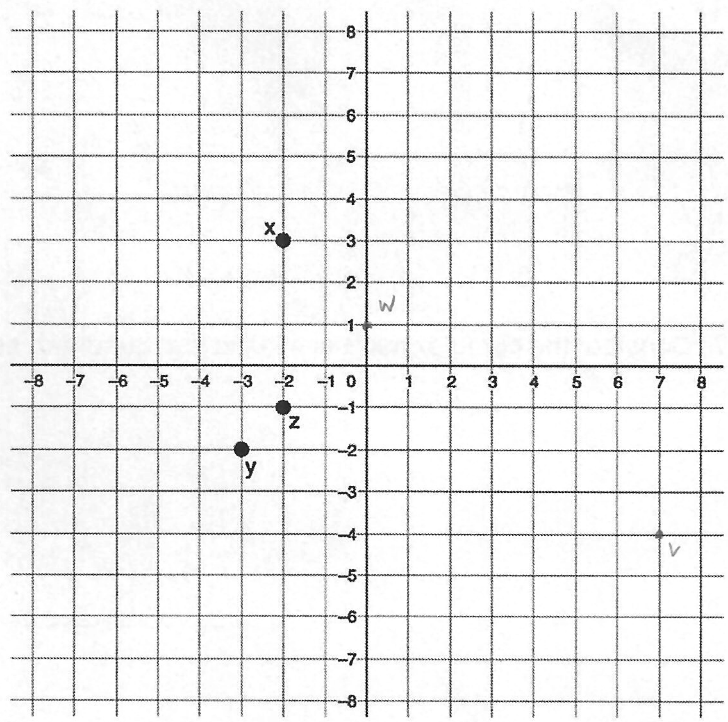
5. Consider the complex numbers $x = -2 + 3i$, $y = -3 - 2i$, and $z = -2 - i$ shown on the right. [2 pts each]

−1 a) Find the matrix of the transformation that maps x to y.

$(-2, 3) \to (-3, 2)$

$T\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



b) Find a complex number w such that wx = y. Then draw and label w on the coordinate axes to the right.

w = i ✓ (in a + bi form)

c) Find a complex number v such that zx = v. Then draw and label v on the coordinate axes to the right.

v = $7 - 4i$ ✓ (in a + bi form)

d) Find the matrix of the transformation that maps x to v.

$(-2, 3) \to (7, -4)$

$T\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

$T = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$ ✓

$zx = v$

$(-2 - i)(-2 + 3i) = v$

$4 - 6i + 2i - 3i^2 = v$

$v = 7 - 4i$

$wx = y$

$w(-2 + 3i) = -3 - 2i$

$w = \frac{-3 - 2i}{-2 + 3i}\left(\frac{-2 - 3i}{-2 - 3i}\right)$

$w = \frac{6 + 9i + 4i + 6i^2}{4 - 9i^2}$

$= \frac{13i}{13}$

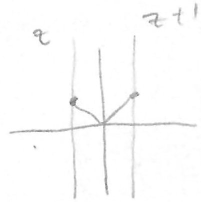
$w = i$

−3

6. Consider the equation $z^4 = (z+1)^4$

a) Explain, using words, vector diagrams, and/or DeMoivre's Theorem, why z **cannot** be in the 1st quadrant. [3 pts]

$|z^4| = |(z+1)^4| \rightarrow |z| = |z+1|$



z and z+1 must be 0.5 away from the complex axis in order for its magnitude to be equal. Since z+1 moves z to the positive real axis, Re(z) must be -0.5, therefore z cannot be in 1st Q ✓

-2 b) There are 3 possible answers for z. Find all of them. Give your answers in $a + bi$ form. [3 pts]

$r = \sqrt{a^2+b^2}$

$r = \sqrt{0.5^2+b^2}$

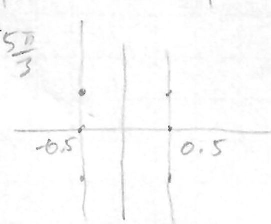
$cis\frac{2\pi}{3}, \frac{5\pi}{3}$

$\cos = -\frac{1}{2}$

$\sin = \frac{\sqrt{3}}{2}$

rotate $\frac{\pi}{2}$

$r = 1$



z^4 and $(z+1)^4$ has to return to the same place → period of 4.

0.5

$z = -0.5$ $\quad z+1 = 0.5$

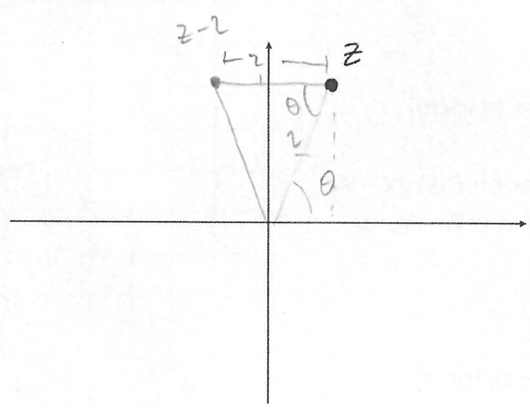
$z^4 = (-0.5)^4 = 0.5^4$ $\quad (z+1)^4 = 0.5^4$

$z = -0.5$ ✓

$z = -0.5 + \frac{\sqrt{7}}{2}i$ /

$z = -0.5 - \frac{\sqrt{3}}{2}i$ /

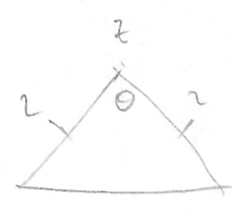
7. Consider the complex number z in the first quadrant, as shown in the diagram below. $|z| = 2$ and $Arg(z) = \theta$.



✓

a) Draw $(z-2)$ onto the diagram, and draw the triangle that is created by the origin, z, and $(z-2)$. [3 pts]

b) Find an expression for the area of the triangle, in terms of θ. [4 pts]



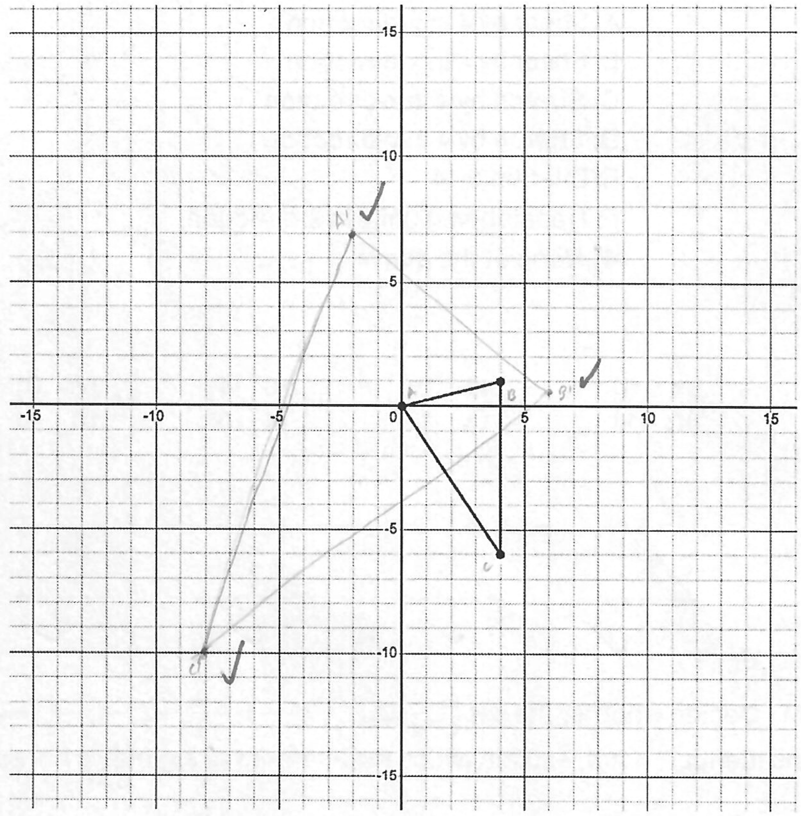
Area $S = \frac{1}{2}(2)(2)\sin\theta$

$S = 2\sin\theta$ ✓

-2

8. Given the transformation matrix T below and the pre-image graphed on the right,

$$T = \begin{bmatrix} 1.5 & 2 & -2 \\ -2 & 1.5 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$



a) Apply the transformation to the points and graph the image on the same axis. [3 pts]

b) (**multiple choice**: circle the BEST answer) The transformation given by matrix T is: [2 pts]

i) a reflection, a dilation, and a shear (in that order)

ii) a rotation, a stretch, and a shear (in that order)

iii) a rotation, a dilation, and a translation (in that order)

iv) a reflection, a dilation, and a translation (in that order)

9. The maximum period of an n-post snap group is 105. Find n. Justify your answer. [3 pts]

lcm (3, 5, 7) = 105

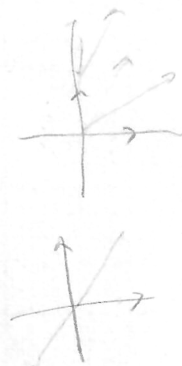
15 post snap group

3+5+7=15.

max period of 26 post is larger than 105.

10. For each matrix below, write a letter A-G that best describes the corresponding transformation. [1 pt each]

A: Shear by 4 in y-direction
B: Shear by 4 in x-direction
C: Stretch by 4 in x-direction
D: Stretch by 4 in y-direction
E: Dilation by 4
F: Translation 4 units in x-direction
G: None of the above



i) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ iii) $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ iv) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ v) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ vi) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C ✓ A ✓ G ✓ E ✓ G ✓ F ✓

11. Decide whether the set $\left\{1, \frac{1}{2}, 2, i, \frac{i}{2}, 2i, -1, -\frac{1}{2}, -2, -i, -\frac{i}{2}, -2i\right\}$ with multiplication is a group. If yes, what is the identity? If not, explain which requirement of a group is not satisfied. [2 pts]

Not a group. No closure. $\frac{1}{2} \cdot \frac{1}{2} \to \frac{1}{4}$ which is not in group.
$2 \cdot 2i = 4i$ " ✓

12. Decide whether the set of all real numbers under the binary operation defined as **a ★ b = a + b – ab** is a group. If yes, what is the identity? If not, explain which requirement of a group is not satisfied. [2 pts]

ID: 0
Inverse: $1 ★ b = 0$
$1 + b - b = 0$
$1 = 0$ no inverse ✓

$2 ★ b = 0$
$2 + b - 2b = 0$
$2 - b = 0$
$b = 2$
$2^{-1} = 2$

$3 ★ b = 0$
$3 + b - 3b = 0$

Not a group.
1 doesn't have an Inverse.

-0