Ch 9: Parametric, Polar and Vector Calculus: Review Sheet

No calculator except as marked. The Quiz itself will have 2 calculator pages, and 1 non-calculator page.

1. If $x = t^2$ and $y = \ln(t^2+1)$, $\dfrac{d^2y}{dx^2}$ is $\underline{-\dfrac{1}{(t^2+1)^2}}$ in terms of t.

$\frac{dx}{dt} = 2t$ $\quad \frac{dy}{dt} = \frac{2t}{t^2+1}$

$\frac{dy}{dt} = \frac{\frac{2t}{t^2+1}}{2t} = \frac{1}{t^2+1}$

$\frac{d}{dx}\left(\frac{1}{t^2+1}\right) = \frac{d}{dt}(t^2+1)^{-1} \cdot \frac{dt}{dx}$

$= -1(t^2+1)^{-2} \cdot 2t \div \frac{dx}{dt}$

$= \frac{-1 \cdot 2t}{(t^2+1)^2} \cdot \frac{1}{2t}$

2. For any time $t \ge 0$, if the position of a particle in the *xy*-plane is given by $x = e^t$ and $y = e^{-t}$, then the speed of the particle at time $t = 1$ is $\underline{2.743}$

speed: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{e^2 + \frac{1}{e^2}} =$ $\quad \frac{dx}{dt} = e^t$ $\quad \frac{dy}{dt} = -e^{-t}$

3. An integral expression for the length of the curve determined by the parametric equations $x = \sin t$ and $y = t$ from $t = 0$ to $t = \pi$ is ____________________.

$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\frac{dx}{dt} = \cos t$ $\quad \frac{dy}{dt} = 1$ $\quad \int_0^\pi \sqrt{\cos^2 t + 1^2}\, dt$

4. Consider the curve in the *xy*-plane represented by $x = \dfrac{2}{t}$ and $y = \ln t$ for $t > 0$. The slope of the tangent line to the curve at the point where $x = 1$ is __________. -1

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ at $t = 2$

$x = 1$, then $t = 2$

$x = 2t^{-1}$

$\left.\begin{aligned} \frac{dy}{dt} &= \frac{1}{t} \\ \frac{dx}{dt} &= \frac{-2}{t^2} \end{aligned}\right\}$ at $t = 2$ $\quad \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$

5. A curve is defined parametrically by $x = e^t$ and $y = 2e^{-t}$. An equation of the tangent line to the curve at $t = \ln 2$ is $\underline{y = -\frac{1}{2}(x-2)+1}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2e^{-t}}{e^t} = \frac{-2}{e^{2t}}$

$x = 2$ $\quad y = 2e^{-\ln 2}$

$y = \frac{2}{e^{\ln 2}} = \frac{2}{2} = 1$

at $t = \ln 2$ $\quad \frac{-2e^{-\ln 2}}{e^{\ln 2}} = \frac{-2 \cdot 2^{-1}}{2} = -\frac{1}{2}$

6. The position of a particle moving in the *xy*-plane is given by $x = t^2 + 2t$, $y = 2t^2 - 6t$. When $t = 2$, the speed of the particle is $\underline{2\sqrt{10}}$.

$x'(t) = 2t+2$ $\quad y'(t) = 4t-6$

speed $= \sqrt{[x'(t)]^2 + [y'(t)]^2}$

$x'(2) = 6$ $\quad y'(2) = 2$

$\sqrt{6^2+2^2} = \sqrt{40}$

7. If $x = \sin t$ and $y = \cos^2 t$, then $\dfrac{d^2y}{dx^2}$ is at $t = \dfrac{\pi}{2}$ is $\underline{-2}$

$\frac{dy}{dt} = -2\cos t \sin t$ $\quad \frac{dy}{dx} = \frac{-2\cos t \sin t}{\cos t} = -2\sin t$

$\frac{dx}{dt} = \cos t$

$\frac{d}{dx}[-2\sin t] = \frac{d}{dt}[-2\sin t] \cdot \frac{dt}{dx}$

$= -2\cos t \cdot \frac{1}{dx/dt}$ $\quad = -2\cos t \cdot \frac{1}{\cos t}$

8. A curve is parametrically defined by the equations $x = 2\cos t$ and $y = 2\sin t$. The length of the arc from $t = 0$ to $t = 2$ is given by the integral ____________________.

$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

$\int_0^2 \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt$

$= \int_0^2 \sqrt{4}\, dt = 2t\Big|_0^2 = 4$

9. If a particle moves in the *xy*-plane so that at time $t > 0$ its position vector is $\left\langle \sin\left(3t - \frac{\pi}{2}\right), 3t^2 \right\rangle$, then at time $t = \frac{\pi}{2}$ the velocity vector is $\langle 0, 3\pi \rangle$ and the speed is 3π, and the acceleration vector is ________________.

$v(t) = r'(t) = \langle 3\cos(3t - \pi/2), 6t \rangle$

speed $= \sqrt{0^2 + (3\pi)^2} = 3\pi$

$a(t) = v'(t) = \langle -9\sin(3t - \pi/2), 6 \rangle$

10. A particle moves on the circle $x^2 + y^2 = 1$ so that at time $t \geq 0$ its position is given by the vector $\mathbf{r}(t) = \left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right\rangle$. As t increases without bound, the coordinates of the point that the particle approaches $(-1, 0)$. *remember notation $\lim_{t\to\infty} \frac{1-t^2}{1+t^2} = -1$ $\lim_{t\to\infty} \frac{2t}{1+t^2} = 0$

11. A curve C is defined by the parametric equations $x = \frac{1}{\sqrt{t+1}}$ and $y = \frac{t}{t+1}$ for $t \geq 0$.

An equation for the curve C in terms of x and y is $y = 1 - x^2$.

$x = \frac{1}{\sqrt{t+1}}$ $\quad t + 1 = \frac{1}{x^2}$

$\sqrt{t+1} = \frac{1}{x}$ $\quad t = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2}$

$y = \left(\dfrac{\frac{1-x^2}{x^2}}{\frac{1-x^2}{x^2} + 1} \right) \frac{x^2}{x^2} = \frac{1-x^2}{1-x^2+x^2} = \frac{1-x^2}{1}$

12. Write a set of parametric equations which would graph the polar curve with the equation $r = 2 - 3\cos\theta$.

$x(\theta) = 2\cos\theta - 3\cos^2\theta$

$y(\theta) = 2\sin\theta - 3\cos\theta\sin\theta$

13. Suppose the function $r = f(\theta)$ is differentiable. The table gives values for f and f' at $\theta = \frac{\pi}{2}$.

$y = f(\theta)\sin\theta$

$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$

θ	$f(\theta)$	$f'(\theta)$
$\frac{\pi}{2}$	10	4

$x = f(\theta)\cos\theta$

$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$

The slope of the line tangent to the graph of $r = f(\theta)$ at the point where $\theta = \frac{\pi}{2}$ is $-2/5$.

$\frac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ at $\frac{\pi}{2}$ $= \dfrac{4\sin\pi/2 + 10\cdot\cos\pi/2}{4\cos\pi/2 - 10\sin\pi/2} = \dfrac{4}{-10}$

$y(\theta) = r\sin\theta$ $\quad r(\frac{\pi}{6}) = 1+\sqrt{3}/2 \quad r'(\theta) = -\sin(\pi/6) = -1/2$

$y'(\theta) = r'\sin\theta + r\cos\theta$

$y'(\pi/6) = -\frac{1}{2}\cdot\frac{1}{2} + (1+\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2}) = -\frac{1}{4} + \frac{2\sqrt{3}+3}{4} = \frac{\sqrt{3}+1}{2}$

14. The slope of the tangent line to the cardioid $r = 1+\cos\theta$ at $\theta = \dfrac{\pi}{6}$ is ____ -1 ____.

$x(\theta) = r\cos\theta$

$x'(\theta) = r'\cos\theta - r\sin\theta$

$x'(\pi/6) = -\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{2+\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{-\sqrt{3}-2-\sqrt{3}}{4} = \frac{-1-\sqrt{3}}{2}$

$\frac{dy}{dx} = \frac{\sqrt{3}+1}{2}\cdot\frac{2}{(-1-\sqrt{3})} = -1$

15. Consider the polar equation $r = 2-2\cos\theta$. $\frac{dr}{d\theta} = 2\sin\theta$

a) Express the derivative, $\dfrac{dy}{dx}$, for the given polar equation. Try to simplify

$y(\theta) = r\sin\theta$

$\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta$

$= 2\cos\theta - 2(\cos^2\theta - \sin^2\theta)$

$= 2\cos\theta - 2\cos 2\theta$

$x(\theta) = r\cos\theta$

$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta = 2\sin\theta\cos\theta - 2\sin\theta + 2\sin\theta\cos\theta$

$= -2\sin\theta + 2\sin 2\theta$

$\frac{dy}{dx} = \frac{2\cos\theta - 2\cos 2\theta}{-2\sin\theta + 2\sin 2\theta}$

b. Find all values of θ for which the polar graph has only a vertical tangent line. $\rightarrow$ at $x = \pi, \pi/3, \frac{5\pi}{3}$

$\frac{dx}{d\theta} = 0$ but $\frac{dy}{d\theta} \neq 0$

$-2\sin\theta + 4\sin\theta\cos\theta = 0$

$2\sin\theta(1 - 2\cos\theta) = 0$

$\sin\theta = 0 \quad \cos\theta = 1/2$

$\theta = \varnothing, \pi, \pi/3, 5\pi/3$

$2\cos\theta - 2\cos 2\theta = 0$

$\cos\theta - \cos 2\theta = 0$

$\cos\theta - (2\cos^2\theta + 1) = 0$

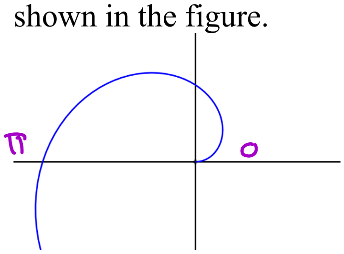
$0 = 2\cos^2\theta - \cos\theta - 1$

$0 = (2\cos\theta + 1)(\cos\theta - 1)$

$\cos\theta = 1 \; ; \; \cos\theta = -1/2$

$\theta = \varnothing, \; 2\pi/3, 4\pi/3$

16. Find the area of the region enclosed by the polar curve $r = \theta$ and the horizontal axis as shown in the figure.



$\int_0^{\pi} \frac{1}{2} r^2 \, d\theta$

$\int_0^{\pi} \frac{1}{2} \theta \, d\theta$

$\frac{1}{2}\cdot\frac{1}{2}\theta^2 \Big|_0^{\pi}$

$A = \frac{1}{4}\pi^2$

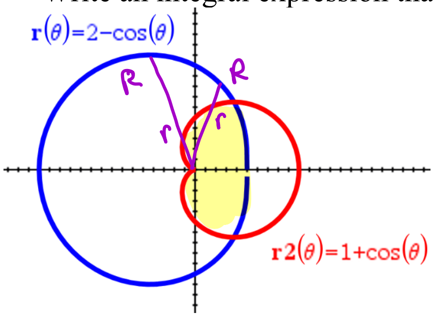
Note:
arclength of a polar function can be found with

$\int_a^b \sqrt{r^2 + [r'(\theta)]^2} \, d\theta$

For 17 – 19: Use your calculator to check if your integral is correct. Make sure that your calculator is in radian mode, and that limits of integration are given at least to 3 decimal places.

17. Let R be the region that lies inside the polar curves $r = 2 - \cos\theta$ and inside the polar curve $r = 1 + \cos\theta$.

Write an integral expression that results in the area of this region.



$2 - \cos\theta = 1 + \cos\theta$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \pi/3,\ 5\pi/3$

$$\int_{\pi/3}^{5\pi/3} \frac{1}{2}(2-\cos\theta)^2 - \frac{1}{2}(1+\cos\theta)^2\, d\theta$$

18. Find the area of the region that lies inside both the curve $r_2 = \cos\theta$ and $r_1 = \sin\theta$.

$\cos\theta = \sin\theta$

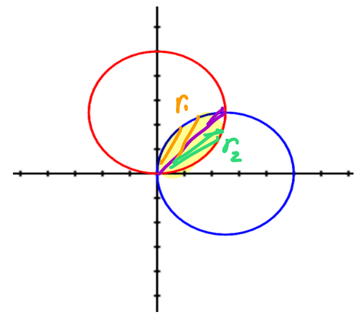
at $\theta = \pi/4$ and $\frac{5\pi}{4}$

$$\int_0^{\pi/4} \frac{1}{2}\cos^2\theta + \frac{1}{2}\sin^2\theta\, d\theta$$

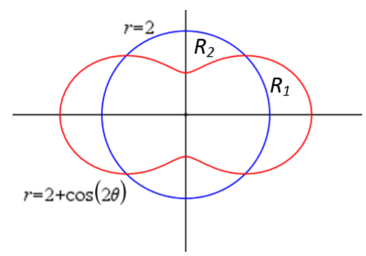
$$\int_0^{\pi/4} \frac{1}{2}\, d\theta$$

$$\frac{1}{2}\cdot\theta\Big|_0^{\pi/4}$$

$\pi/8$



19. The figure to the right shows the graphs of the polar curves $r = 2$ and $r = 2 + \cos(2\theta)$. Let R_1 be the shaded region in the first quadrant bounded by the two curves and the horizontal axis, and R_2 be the shaded region in the first quadrant bounded by the two curves and the vertical axis.



a. Set up, but do not evaluate, an integral expression that represent the area of region R_1.

$2 = 2 + \cos 2\theta$

$0 = \cos 2\theta$

$2\theta = \pi/2,\ 3\pi/2$

$\theta = \pi/4,\ 3\pi/4,\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

$$R_1 = \int_0^{\pi/4} \frac{1}{2}(2+\cos 2\theta)^2 - \frac{1}{2}\cdot 2^2\, d\theta$$

b. Set up, but do not evaluate, an integral expression that represents the area of region R_2.

$$R_2 = \int_{\pi/4}^{\pi/2} \frac{1}{2}\cdot 2^2 - \frac{1}{2}(2+\cos 2\theta)^2\, d\theta$$