FRQ Practice 2: Vector, Parametric, Volume, Area, Integration strategies

2011

A graphing calculator is required for these problems.

1. At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin(t^2)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

   (a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

   (b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

   (c) Find the position of the particle at time $t = 3$.

   (d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

Calculator allowed: 2008

1. Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the vertical line $x = -1$.

   (c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the y-axis are squares. Find the volume of this solid.

Add on for practice:

(d) Find the volume of the solid when region R is rotated around the vertical line x = 10

No Calculator is permitted for these problems.

2016

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

   (a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

   (b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

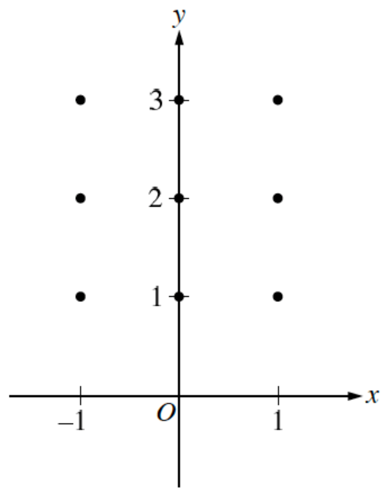
   (c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

   (d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

1998

4. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

(a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

Add on for practice: (d) Find a set of ordered pairs where the particular solution to this differential equation would be constant.

(e) Make a conjecture about the locations of points of inflection for all non-constant solution curves.

2022

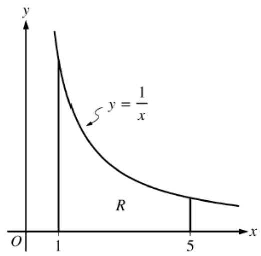


Figure 1

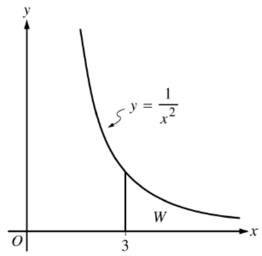


Figure 2

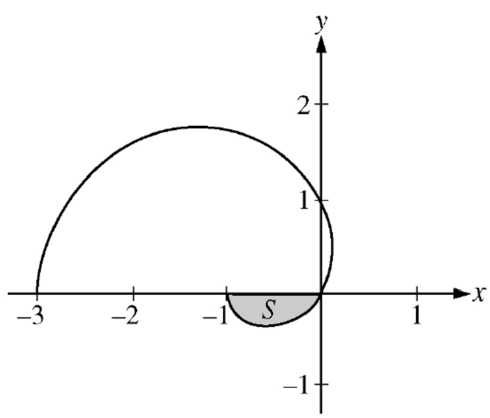
5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

2009



4. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

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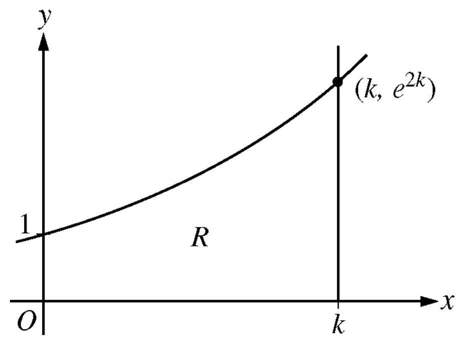
Question 3

Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes, and the vertical line $x = k$, where $k > 0$. The region R is shown in the figure above.



(a) Write, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

(b) The region R is rotated about the x-axis to form a solid. Find the volume, V, of the solid in terms of k.

(c) The volume V, found in part (b), changes as k changes. If $\frac{dk}{dt} = \frac{1}{3}$, determine $\frac{dV}{dt}$ when $k = \frac{1}{2}$.

(a) $f'(x) = 2e^{2x}$

Perimeter $= 1 + k + e^{2k} + \int_0^k \sqrt{1 + \left(2e^{2x}\right)^2}\, dx$

$$3 : \begin{cases} 1 : f'(x) \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(b) Volume $= \pi \int_0^k \left(e^{2x}\right)^2 dx = \pi \int_0^k e^{4x}\, dx = \frac{\pi}{4} e^{4x} \Big|_{x=0}^{x=k} = \frac{\pi}{4} e^{4k} - \frac{\pi}{4}$

$$4 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

(c) $\frac{dV}{dt} = \pi e^{4k} \frac{dk}{dt}$

When $k = \frac{1}{2}$, $\frac{dV}{dt} = \pi e^2 \cdot \frac{1}{3}$.

$$2 : \begin{cases} 1 : \text{applies chain rule} \\ 1 : \text{answer} \end{cases}$$

Question 1

Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the vertical line $x = -1$.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the y-axis are squares. Find the volume of this solid.

The graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$ intersect at the points $(0, 0)$ and $(9, 3)$.

(a) $\displaystyle\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = 4.5$

OR

$\displaystyle\int_0^3 \left(3y - y^2\right) dy = 4.5$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $\displaystyle\pi\int_0^3 \left((3y+1)^2 - \left(y^2+1\right)^2\right) dy$

$\displaystyle = \frac{207\pi}{5} = 130.061 \text{ or } 130.062$

$4 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\displaystyle\int_0^3 \left(3y - y^2\right)^2 dy = 8.1$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

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Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$	$2 : \frac{d^2y}{dx^2}$ in terms of x and y
(b) $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$ $\left.\frac{d^2y}{dx^2}\right\rvert_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$ Thus, the graph of f has a relative maximum at the point $(-2, 8)$.	2 : conclusion with justification
(c) $\lim_{x \to -1}(g(x) - 2) = 0$ and $\lim_{x \to -1} 3(x+1)^2 = 0$ Using L'Hospital's Rule, $\lim_{x \to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) = \lim_{x \to -1}\left(\frac{g'(x)}{6(x+1)}\right)$ $\lim_{x \to -1} g'(x) = 0$ and $\lim_{x \to -1} 6(x+1) = 0$ Using L'Hospital's Rule, $\lim_{x \to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x \to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$	$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$
(d) $h\left(\frac{1}{2}\right) \approx h(0) + h'(0)\cdot\frac{1}{2} = 2 + (-1)\cdot\frac{1}{2} = \frac{3}{2}$ $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\cdot\frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{5}{4}$	$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

1998 Calculus BC Scoring Guidelines

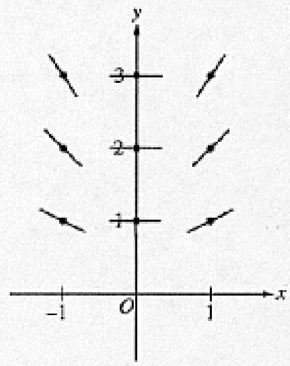
4. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

(a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

(a)



1: line segments at nine points with negative - zero - positive slope left to right and increasing steepness bottom to top at $x = 1$ and $x = -1$

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$$= 3 + \frac{1}{2}(0)(3)(0.1) = 3$$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$$= 3 + \frac{1}{2}(0.1)(3)(0.1)$$

$$= 3 + \frac{.03}{2} = 3.015$$

2 $\begin{cases} 1\text{: Euler's Method equations or equivalent table} \\ 1\text{: answer (not eligible without first point)} \end{cases}$

Special Case: 1/2 for first iteration 3.015 and second iteration 3.045

(c) $\frac{dy}{dx} = \frac{xy}{2}$

$$\int \frac{dy}{y} = \int \frac{x}{2}\, dx$$

$$\ln|y| = \frac{1}{4}x^2 + C_1$$

$$y = Ce^{x^2/4}$$

$$3 = Ce^0 \implies C = 3$$

$$y = 3e^{x^2/4}$$

$$f(0.2) = 3e^{.04/4} = 3e^{.01} = 3.030$$

6 $\begin{cases} 1\text{: separates variables} \\ 1\text{: antiderivative of } dy \text{ term} \\ 1\text{: antiderivative of } dx \text{ term} \\ 1\text{: solves for } y \\ 1\text{: solves for constant of integration} \\ 1\text{: evaluates } f(0.2) \end{cases}$

Note: max 4/6 [1-1-1-0-0-1] if no constant of integration

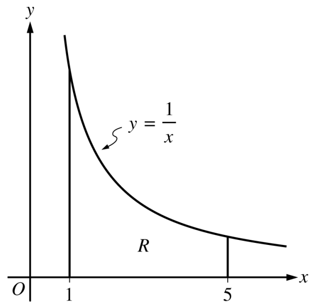


Figure 1

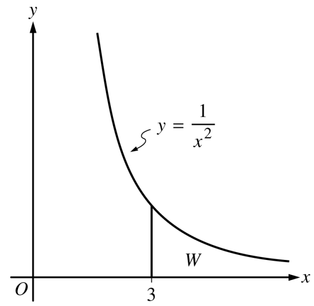


Figure 2

5. Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

(a) Find the area of region R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

Model Solution	Scoring
(a) Area $= \int_1^5 \frac{1}{x}\,dx = \ln x\Big\vert_1^5 = \ln 5 - \ln 1 = \ln 5$	$2: \begin{cases} 1: \text{Integral} \\ 1: \text{Answer} \end{cases}$
(b) Volume $= \int_1^5 xe^{x/5}\,dx$ $u = x$ $dv = e^{x/5}\,dx$ $du = dx$ $v = 5e^{x/5}$ $\int xe^{x/5}\,dx = 5xe^{x/5} - \int 5e^{x/5}\,dx$ $= 5xe^{x/5} - 25e^{x/5} + C = 5e^{x/5}(x-5) + C$ Volume $= 5e^{x/5}(x-5)\Big\vert_1^5 = 5e(0) - 5e^{1/5}(-4) = 20e^{1/5}$	$4: \begin{cases} 1: \text{Definite integral} \\ 1: u \text{ and } dv \\ 1: \int xe^{x/5}dx \\ \qquad = 5xe^{x/5} - \int 5e^{x/5}dx \\ 1: \text{Answer} \end{cases}$
(c) Volume $= \pi \int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty} \int_3^b \frac{1}{x^4}\,dx = \pi \lim_{b\to\infty} \left(\frac{1}{-3x^3}\Big\vert_3^b\right)$ $= \pi \lim_{b\to\infty} \left(\frac{1}{-3}\right)\left[\frac{1}{b^3} - \frac{1}{3^3}\right] = \pi\left(\frac{1}{-3}\right)\left(0 - \frac{1}{3^3}\right) = \frac{\pi}{81}$	$3: \begin{cases} 1: \text{Improper integral} \\ 1: \text{Antiderivative} \\ 1: \text{Answer} \end{cases}$

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

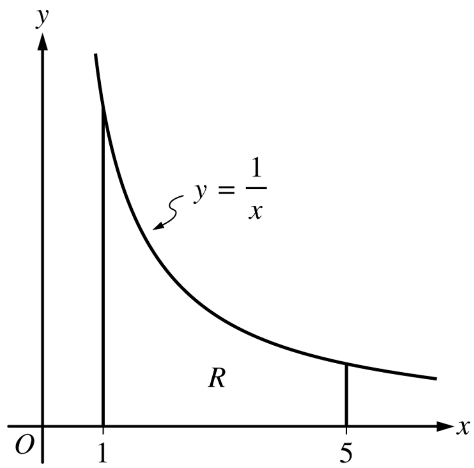


Figure 1

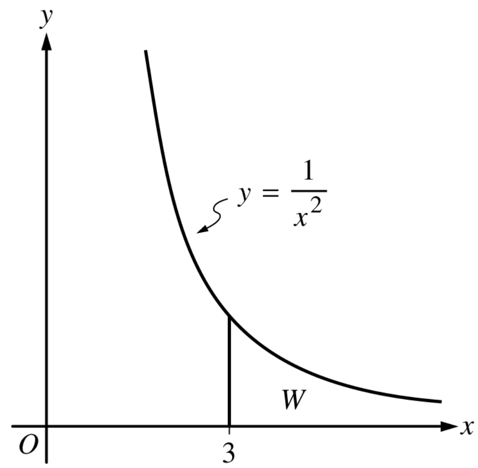


Figure 2

Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

Model Solution	Scoring	
(a) Find the area of region R.		
$\text{Area} = \int_1^5 \frac{1}{x}\,dx$	Integral	**1 point**
$= \ln x \Big\vert_1^5$ $= \ln 5 - \ln 1 = \ln 5$	Answer	**1 point**

Scoring notes:

- A definite integral with incorrect bounds does not earn either point.
- An unevaluated indefinite integral does not earn either point.
- An indefinite integral which is evaluated in a later step may earn one or both points. For example, $\int \frac{1}{x}\,dx = \ln 5 - \ln 1$ (or $\ln 5$) does not earn the first point but does earn the second. However, $\int \frac{1}{x}\,dx = \ln x + C \Rightarrow \text{Area} = \ln 5 - \ln 1$ earns both points.

Total for part (a) 2 points

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

$\text{Volume} = \int_1^5 xe^{x/5}\,dx$	Definite integral	**1 point**
Using integration by parts, $u = x \qquad dv = e^{x/5}\,dx$ $du = dx \quad v = 5e^{x/5}$	u and dv	**1 point**
$\int xe^{x/5}\,dx = 5xe^{x/5} - \int 5e^{x/5}\,dx$ $= 5xe^{x/5} - 25e^{x/5} + C$ $= 5e^{x/5}(x-5) + C$	$\int xe^{x/5}\,dx$ $= 5xe^{x/5} - \int 5e^{x/5}\,dx$	**1 point**
$\text{Volume} = 5e^{x/5}(x-5)\Big\|_1^5$ $= 5e(0) - 5e^{1/5}(-4) = 20e^{1/5}$	Answer	**1 point**

Scoring notes:

- The first point is earned for $c\int_1^5 xe^{x/5}\,dx$, where $c \neq 0$. Errors of $c \neq 1$, for example $c = \pi$, will not earn the fourth point.
- Incorrect integrals that require integration by parts are still eligible for the second and third points. Both of these points will be earned with at least one correct application of integration by parts.
- The second point will be earned with an implied u and dv in the presence of $5xe^{x/5} - \int 5e^{x/5}\,dx$.
- The tabular method may be used to show integration by parts. In this case, the second point is earned by having columns (labeled or unlabeled) that begin with x and $e^{x/5}$. The third point is earned for either $5xe^{x/5} - \int 5e^{x/5}\,dx$ or $5xe^{x/5} - 25e^{x/5}$.
- Limits of integration may be present, omitted, or partially present in the work for the second and third points.
- The fourth point is earned only for the correct answer.

Total for part (b) 4 points

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

$\text{Volume} = \pi\int_3^{\infty}\left(\frac{1}{x^2}\right)^2 dx = \pi\lim_{b\to\infty}\int_3^b \frac{1}{x^4}\,dx$	Improper integral	**1 point**
$= \pi\lim_{b\to\infty}\left(\left.\frac{1}{-3x^3}\right\rvert_3^b\right)$	Antiderivative	**1 point**
$= \pi\lim_{b\to\infty}\left(\frac{1}{-3}\right)\left[\frac{1}{b^3}-\frac{1}{3^3}\right]$ $= \pi\left(\frac{1}{-3}\right)\left(0-\frac{1}{3^3}\right) = \frac{\pi}{81}$	Answer	**1 point**

Scoring notes:

- The first point is earned for either $c\int_3^{\infty}\left(\frac{1}{x^2}\right)^2 dx$ or $\lim_{b\to\infty} c\int_3^b \frac{1}{x^4}\,dx,$ where $c \neq 0.$ Errors of $c \neq \pi$ will not earn the third point.
- The second point is earned for a correct antiderivative of any integrand of the form $\frac{1}{x^n},$ for any integer $n \geq 2.$
- To earn the answer point, a response must use correct limit notation and cannot include arithmetic with infinity such as $\frac{1}{\infty^3}.$

Total for part (c) 3 points

Total for question 5 9 points

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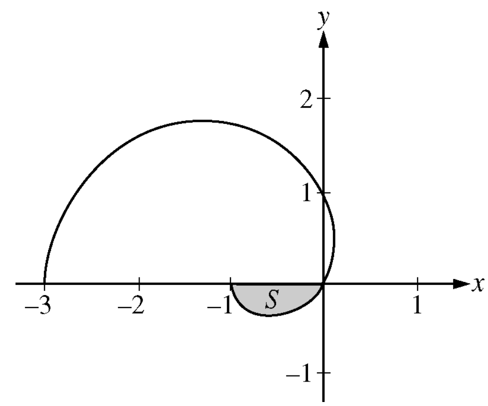
Question 4



The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

(a) $r(0) = -1;\ r(\theta) = 0$ when $\theta = \frac{\pi}{3}$.

Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$$

$4 : \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \frac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$

(c) When $\theta = \frac{\pi}{2}$, we have $x = 0,\ y = 1$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$$

The tangent line is given by $y = 1 - 2x$.

$3 : \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \frac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$